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E. A. MILNE, E. C. TITCHMARSH

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COMMUTATIVE FUNCTIONS (I)

By A. G. WALKER (*Liverpool*)

[Received 25 July 1944]

1. Introduction

COMMUTATIVE functions, i.e. functions such as f, g satisfying

$$f\{g(x)\} \equiv g\{f(x)\},$$

arose in Milne's *Kinematical Relativity*, and their canonical forms under certain circumstances have important consequences in this theory. G. J. Whitrow* gave many useful and interesting properties, but no detailed study has yet been made of such functions, which have been found increasingly important in more recent work.† The object of this paper is to present such a study, and a number of significant theorems on *related functions* and the *canonical form* are given in Part I.

It is believed that commutative functions will also be found interesting in themselves, and I do not hesitate at times to pursue lines of thought which do not necessarily lead to applicable results. This is particularly noticeable in Part II, which is concerned with *semi-related functions*, these being interesting by-products of the search for canonical forms.

2. Notation and definitions

Most functions considered here are of one variable and are *c.m.i.*, this being written for 'continuous and (strictly) monotonic increasing'.

The *functional product* of f and g is the function $f\{g(x)\}$, which will be written $fg(x)$, or simply fg. In general fg and gf are not identical, i.e. f and g are not commutative.

The associative law holds for functional products, thus

$$f(gh) = (fg)h = fgh.$$

For n positive integer n, f^n is defined by

$$f^2 = ff, \qquad f^3 = ff^2, \quad \ldots, \quad f^n = ff^{n-1}.$$

* (1) *Quart. J. of Math.* (Oxford), 6 (1935), 249–60, (2) *Zeits. f. Astrophys.* 12 (1936), 47–55, and (3) *Zeits. f. Astrophys.* 15 (1938), 263–98, the latter with E. A. Milne. See also W. H. M'Crea, *Proc. Roy. Irish Academy*, 45 (A) (1938), 23–30.

† A. G. Walker, 'Foundations of Relativity.' (Not yet published.)

It is clear that the above products exist only when the intervals of definition and the ranges of values of the given functions are suitable. The necessary conditions will not be considered now, but the matter must not be forgotten later.

If f, g, h are c.m.i., then the above products also have this property. Further, inverse functions f^{-1}, etc., exist and are c.m.i., and we have the formula

$$(fgh\ldots)^{-1} = \ldots h^{-1}g^{-1}f^{-1}.$$

For a positive integer n, f^{-n} is defined as $(f^{-1})^n$, and for any integers* p, q we have

$$f^p f^q = f^q f^p = f^{p+q}, \qquad (f^p)^q = f^{pq}.$$

When a function is being considered throughout an interval, a single symbol will usually be used, and statements such as $f = g$, $fg = gf$, $f > g$, denote relations existing throughout an interval. The identity function is denoted by I, so that $I(x) \equiv x$, and the function αx (α constant) is denoted by α when there is no ambiguity, so that $f\alpha$, αf are the functions $f(\alpha x)$, $\alpha f(x)$ respectively.

Definitions. Functions f, g will be said to *intersect* at x_0 if $f(x_0) = g(x_0)$.

A *node* of f is a number satisfying $f(x) = x$, i.e. an intersection of f and I.

A function f is said to be *complete* in an interval† $\langle a, b\rangle$ if it is c.m.i. and if a, b are nodes of f.

In the study of commutative functions and canonical forms we shall consider only functions complete in some interval. For it can be shown‡ that functions not complete in a given interval can be continued over an extended interval, after a transformation if necessary, so that they become complete. This does not affect their properties in the original interval, now a sub-interval, and we get the advantage of greater precision, since it becomes unnecessary to consider the various possible end-conditions for incomplete functions.

* Throughout this paper m, n will denote positive integers, and p, q, r, s, u, v will denote integers which may be positive, negative, or zero; $f^0(x)$ is simply x.

† Closed and semi-closed intervals are written $\langle a, b\rangle$, and $\langle a, b)$, $(a, b\rangle$. It is convenient to treat symbols ∞, $-\infty$, as if they represent numbers. Thus, for example, the function $f(x) \equiv \alpha x$ ($\alpha > 0$) is complete in $\langle \infty, -\infty\rangle$ since $f(\pm\infty) = \pm\infty$.

‡ In an appendix given at the end of Part II.

3. The canonical form

The canonical form and certain existence theorems connected with it were first studied by G. J. Whitrow. For our purpose the fundamental theorem can be stated as follows.

[3.1] *A function f which is complete in $\langle a,b\rangle$ and has no interior node can be expressed in the canonical form $\psi^{-1}\alpha\psi$ where α is a constant, ψ is c.m.i. in $\langle a,b\rangle$, and $\psi(a)=0$, $\psi(b)=\infty$. Any value can be chosen for α from the range $\alpha>1$ if $f>I$, and from $0<\alpha<1$ if $f<I$ in (a,b).*

The following proof is essentially that given by Whitrow,* and is given here to illustrate the notation.

Suppose first that† $f>I$. Choose any x_0 in (a,b) and write $x_p=f^p(x_0)$. Then x_n, x_{-n} tend monotonically to b, a respectively when $n\to\infty$, since

$$a<x_p<f(x_p)=x_{p+1}<b,$$

and a, b are the only nodes of f. It follows that each x in (a,b) can be expressed uniquely in the form

$$x=f^p(\xi)\quad(x_0\leqslant\xi<x_1;\ p=0,\pm1,\pm2,\ldots). \tag{1}$$

Now choose $\alpha>1$ and a c.m.i. function $\psi(\xi)$ $(x_0\leqslant\xi\leqslant x_1)$ such that $\psi(x_1)=\alpha\psi(x_0)>0$. Define $\psi(x)$ $(a<x<b)$ by

$$\psi f^p(\xi)=\alpha^p\psi(\xi)\quad(x_0\leqslant\xi<x_1;\ p=\pm1,\pm2,\ldots).$$

Then it is easily verified that ψ is c.m.i. in (a,b) and, continuing to a and b by letting $p\to\pm\infty$, $\psi(a)=0$, $\psi(b)=\infty$. Finally, with x expressed as in (1),

$$\psi f(x)=\psi f^{p+1}(\xi)=\alpha^{p+1}\psi(\xi)=\alpha\psi f^p(\xi)=\alpha\psi(x),$$

whence $f=\psi^{-1}\alpha\psi$ in (a,b) and hence in $\langle a,b\rangle$, as required.

Suppose now $f<I$. Then $f^{-1}>I$, and it has therefore been proved that f^{-1} can be expressed as $\psi^{-1}\beta\psi$ $(\beta>1)$. Hence, $f=\psi^{-1}\alpha\psi$, $\alpha=\beta^{-1}$ $(0<\alpha<1)$. This completes the proof.

Let $f=\psi^{-1}\alpha\psi$ as in Theorem [**3.1**] and consider

$$\Psi=\theta\psi,\qquad\theta(x)\equiv-x^{-1}.$$

Then $$f=\psi^{-1}\alpha\psi=\Psi^{-1}\theta\alpha\theta^{-1}\Psi=\Psi^{-1}\beta\Psi\quad(\beta=\alpha^{-1}).$$

* Loc. cit. (**2**), 49.

† It is to be understood that statements concerning inequalities throughout an interval refer only to the *open* interval except where otherwise stated. The interval will not be mentioned when there is no ambiguity.

This gives an alternative canonical form, Ψ being c.m.i. but with $\Psi(a) = -\infty$, $\Psi(b) = 0$.

Again, let $f = \psi^{-1}\alpha\psi$, and consider

$$\Psi = \phi\psi, \qquad \phi(x) \equiv x^{\lambda} \quad (\lambda > 0).$$

Then $$f = \psi^{-1}\alpha\psi = \Psi^{-1}\phi\alpha\phi^{-1}\Psi = \Psi^{-1}\gamma\Psi \quad (\gamma = \alpha^{\lambda}).$$

The canonical form is thus recovered but with a different constant. Any desired constant in the same range as α is given by some λ.

From its construction it is seen that ψ in **[3.1]** is to a certain extent arbitrary for a given α, and to examine this extent suppose that

$$f = \psi^{-1}\alpha\psi = \Psi^{-1}\alpha\Psi, \qquad \chi = \Psi\psi^{-1}.$$

Then χ is c.m.i. and $\chi(\alpha x) \equiv \alpha\chi(x)$ $(x \geqslant 0)$. From this it is easily seen, by writing $\chi(x) \equiv xF(\log x)$, that

$$\Psi(x) \equiv \psi(x)F\{\log\psi(x)\},$$

where F can be any continuous periodic function, of period $\log\alpha$, such that $e^{x}F(x)$ is monotonic increasing.

The canonical form in $\langle a, b\rangle$ with ψ c.m.i. still exists if f has just one interior node c, provided that $f \lesssim I$ in (a, c) and $f \gtrsim I$ in (c, b), in which case $\psi(a) = -\infty$, $\psi(c) = 0$, $\psi(b) = \infty$. This is seen by fitting together the more usual form in (c, b) and the alternative form in (a, c). In all other cases, however, the canonical form can exist only if ψ is allowed to be discontinuous at some of the interior nodes of f.

4. Commutative functions : elementary properties

Suppose that functions f, g are c.m.i. and satisfy $fg = gf$ in some interval. Then immediately, as in commutative algebra,

$$f^{p}g^{q} = g^{q}f^{p}, \qquad (f^{p}g^{q})(f^{r}g^{s}) = f^{p+r}g^{q+s}, \qquad (f^{p}g^{q})^{r} = f^{pr}g^{qr}$$

for any integers p, q, r, s provided the intervals of definition and the ranges of values are suitable. Thus any continued product of f, f^{-1}, g, g^{-1} can be expressed in the form $f^{p}g^{q}$, and any two such functions are commutative. The set of functions $f^{p}g^{q}$ for variable integers p, q will be denoted by $[f, g]$. It includes f^{p}, g^{q}, and I. It is easily verified that the sets $[f, g]$, $[F, G]$ are equivalent if integers p, q, r, s satisfying $ps - qr = \pm 1$ can be found such that

$$F = f^{p}g^{q}, \qquad G = f^{r}g^{s}, \tag{2}$$

The following theorem will now be proved.

[4.1] *If f, g are c.m.i. and commutative in an interval which includes x_0, and if $f^p(x_0) = g^p(x_0)$, then $f(x_0) = g(x_0)$. Hence, if $f^p = g^p$, then $f = g$.*

Writing $h = g^{-1}f$, then $h^p(x_0) = g^{-p}f^p(x_0) = x_0$. If now $h(x_0) > x_0$, then since h^r is c.m.i., we find $h^{r+1}(x_0) > h^r(x_0)$, which leads to $h^p(x_0) \gtrless x_0$ according as $p \gtrless 0$. A similar result is reached by assuming $h(x_0) < x_0$, and we conclude that $h(x_0) = x_0$. Hence, $f(x_0) = gh(x_0) = g(x_0)$ as required.

The following theorem is required later.

[4.2] *If $f_n(x) \to F(x)$, not necessarily uniformly, and if G is continuous and such that $Gf_n = f_n G$ for all n, then $FG = GF$.*

To prove this, we have at once

$$f_n G(x) \to FG(x),$$

and

$$f_n G(x) = Gf_n(x) \to GF(x)$$

since G is continuous.

Another theorem required later is the following:

[4.3] *If f is complete in $\langle a, b\rangle$ and has no interior node, and if g is c.m.i., commutative with f, and has a node in this interval, then g is complete in $\langle a, b\rangle$ and has infinitely many interior nodes.*

To prove this, suppose that t $(a < t < b)$ is a node of g, so that $g(t) = t$. Then (for any integer p) g commutes with f^p and $gf^p(t) = f^p g(t) = f^p(t)$; hence $f^p(t)$ is also a node of g. If $f > I$, then, as in § 3, $f^n(t)$, $f^{-n}(t)$ tend monotonically to b, a respectively as $n \to \infty$, and g therefore has infinitely many nodes including a, b; the argument is similar if $f < I$, the limits a, b being interchanged. The theorem is thus proved in all cases.

A corollary is that, *if f is as defined in* [4.3] *and if g is c.m.i. and commutes with and intersects f, then g is complete and intersects f infinitely often in $\langle a, b\rangle$.* This follows from the fact that $f^{-1}g$ has a node where f and g intersect, so that [4.3] can be applied to f and $f^{-1}g$, every node of the latter now giving an intersection of f and g.

An example of commutative functions is given by

$$f = \phi^u, \qquad g = \phi^v, \tag{3}$$

where ϕ is complete in $\langle a, b\rangle$, and u, v are integers. If u, v are prime to each other, the set $[f, g]$ is equivalent to $[I, \phi]$.

Another example is

$$f = \psi^{-1}\alpha\psi, \qquad g = \psi^{-1}\beta\psi, \tag{4}$$

where α, β are positive constants, and ψ is c.m.i., $\psi(a) = 0, \psi(b) = \infty$. If $\log\alpha/\log\beta$ is a rational number u/v, then (4) is the same as (3) with $\phi = \psi^{-1}\gamma\psi$, $\gamma = \alpha^{1/u} > 0$.

These two examples are the simplest and do not cover all possibilities, as will be seen in the general investigation to which we now turn.

5. Reduction of general case

I proceed to examine the general case of two c.m.i. commutative functions assuming only that they are both complete in some interval $\langle A, B\rangle$. This assumption is not restrictive, for, as mentioned in § 2, the functions can be completed if not already so, in such a way that they remain commutative. (See Part II, appendix.)

Let x_0 be a point in (A, B) other than a node of both functions. Then one of the functions, which we will call f, is complete in a sub-interval $\langle a, b\rangle$ which contains x_0 but no interior node of f, while the other function, g, is complete in $\langle a', b'\rangle$ where $a \leqslant a' < b' \leqslant b$. From [**4.3**] we now see that g is certainly complete in $\langle a, b\rangle$. Since x_0 was any point in (A, B) other than a common node, we have proved the following:

[**5.1**] *$\langle A, B\rangle$ contains one or more intervals in each of which* (i) *both functions are complete and* (ii) *at least one has no interior node. All residual points of $\langle A, B\rangle$ are nodes of both functions.*

If $\langle a, b\rangle$ is one of the above sub-intervals and $a < x < b$, then $a < f^p g^q(x) < b$. Thus values in one such interval are not related to values in another, the commutative relation operating independently in each of the sub-intervals. Hence

[**5.2**] *it is sufficient* to consider commutative functions f, g in an interval in which both are complete and f has no interior node. It can also be assumed that $f > I$ in this interval,* for since $[f, g]$, $[f^{-1}, g]$ are equivalent, f can be replaced by its inverse without affecting the properties we wish to consider, and if $f < I$, then $f^{-1} > I$.

* Sufficient, that is, for the purpose of classification. When commutative functions are examined in the general interval described in [**5.1**], each sub-interval must be considered separately, and the 'class' of the pair of functions may vary from interval to interval.

Supposing now that f is given ($f > I$), let us examine the general form of g subject to [**5.2**]. If we choose x_0 ($a < x_0 < b$) and write $x_1 = f(x_0)$, then g *is arbitrary in* $\langle x_0, x_1 \rangle$ *except that it must be c.m.i. and satisfy*

$$a < g(x_0) < b, \qquad g(x_1) = fg(x_0). \tag{5}$$

For, if x in (a, b) is expressed as in (1), then the definition of g is completed by

$$gf^p(\xi) = f^p g(\xi) \quad (x_0 \leqslant \xi < x_1;\ p = \pm 1, \pm 2, \ldots),$$

and it can now be verified that f, g are as described in [**5.2**].

There are many possible forms of g. For example, g can have one or more nodes in $\langle x_0, x_1 \rangle$, or f and g can intersect one or more times, *these two possibilities being exclusive*. To prove this we first observe that, if t_0 is a node of g, then $t_p = f^p(t_0)$ is also a node, as in the proof of [**4.3**]. Hence, if $t_p < x < t_{p+1}$, then

$$g(x) < g(t_{p+1}) = t_{p+1} = f(t_p) < f(x),$$

showing that f, g cannot intersect in (t_p, t_{p+1}) and so in (a, b). This also implies that if f, g intersect, then g cannot have a node.

It is now possible to see that (3) and (4) cannot include all possibilities. In both examples, f and g can only have nodes together (at a node of ϕ, or at a point at which ψ is zero or infinite), whereas in the above discussion it is clear that g can have nodes alone.

6. Unrelated functions

I shall now classify all pairs of commutative functions subject to [**5.2**], assuming for convenience that $f > I$ in (a, b). It will be shown that there are four distinct classes, and I shall say that the functions are either *related rationally*, *related irrationally*, *semi-related*, or *unrelated*. The first two classes are probably the most important, functions being *related* if they belong to either of them. It is convenient to consider the unrelated class first.

This class contains all the 'intersection' cases. It was seen in § 5 that g may intersect I, or f, or neither, and in the latter case it is possible for g^u and f^v (for some integers u, v) to intersect without being identical. All these cases become similar when viewed in relation to the whole set $[f, g]$, and we say that:

DEFINITION. *Functions f, g are* unrelated *if integers u, v can be found such that f^v, g^u intersect without being identical.*

From Theorem [**4.1**] it can be assumed that $u > 0$, and that u, $|v|$

are prime to each other. The particular cases of intersection mentioned above are given by $u = 1$, $v = 0$, and $u = v = 1$, f^0 being I.

In the general case let integers r, s satisfy

$$ru + sv = 1, \tag{6}$$

and define

$$\theta = f^v g^{-u}, \qquad \phi = f^r g^s. \tag{7}$$

Then, from (2), $[f, g]$ is equivalent to $[\theta, \phi]$, and we find

$$f = \theta^s \phi^u, \qquad g = \theta^{-r} \phi^v. \tag{8}$$

The set $[f, g]$ can be described conveniently in terms of θ and ϕ, which are both members of the set and so complete and commutative in $\langle a, b \rangle$. It will be proved that these functions have the following properties: ϕ has no node in (a, b) while θ has infinitely many, and for any integer p,

$$\phi > \theta^p > \phi^{-1}. \tag{9}$$

For proof, we first see from (7) that the given intersection of f^v, g^u is a node of θ. Therefore, from [4.3], θ has infinitely many nodes in (a, b). For any integer q, θ^q also has these nodes and commutes with f which has no nodes in (a, b). It follows as in § 5 that θ^q cannot intersect f. We now see that ϕ cannot intersect θ^p for any integer p, for otherwise, from (8), f would intersect θ^{s+pu}, which has been proved impossible. Consider now a node X of θ, and therefore of θ^p. Then, from (8), $f(X) = \phi^u \theta^s(X) = \phi^u(X)$, and, since $f > I$ and $u > 0$, it follows that $\phi(X) > X$. Hence $\phi(X) > \theta^p(X)$, and, since $\phi > \theta^p$ holds at one point, it holds throughout (a, b), for otherwise ϕ and θ^p would intersect, which has been proved impossible. Taking inverses it follows immediately that $\phi^{-1} < \theta^p$ for all p, and (9) is established. In particular, $p = 0$ gives $\phi > I$, which excludes ϕ from having a node in (a, b).

Writing $[f, g]_q$ for the sub-set $\theta^p \phi^q$ for fixed q and variable p, it is easily verified that this sub-set is determined uniquely by q, although equation (6) for r, s has many solutions. If now N denotes the set of nodes of θ, then the following properties of $[f, g]$ can easily be verified from (6), (7), (8).

Two functions from different sub-sets do not intersect.

All functions in the same sub-set have a common intersection at each point of N, and no two functions intersect at any other point.

Graphically, the limits of the sub-set $[f, g]_0$ are given by the sides of infinitely many squares diagonally in a straight line, and the

curves corresponding to the functions of this set all lie within these squares. The limits of $[f,g]_q$ are similarly given by a chain of rectangles diagonally along the curve ϕ^q. These rectangles for all q lie within a large square, which they just fill if N consists only of $\phi^p(X)$, X being one point of N.

Examining various cross-sections of $[f,g]$, and using the notation of order-types,* we see that the set of numbers $f^p g^q(x)$, when x is fixed and p, q variable, is of type π if x is a point of N, and π^2 otherwise. The adjacent limit-points in the latter case coincide when and only when the points of N are $\phi^p(X)$ as above.

Before leaving unrelated functions it may be noted that not every function commutative with $\psi^{-1}\alpha\psi$ can be written† $\psi^{-1}F\psi$ for some function F. For, if, in the above notation, $\theta = \psi^{-1}\alpha\psi$ in an interval between consecutive nodes, then $\psi\phi\psi^{-1}$ does not exist, i.e. ϕ cannot be expressed in the form $\psi^{-1}F\psi$. This applies also to most functions of the set $[f,g]$.

7. Rationally related functions

The next class to be considered is the following:

DEFINITION. *Functions f, g are* related rationally *if non-zero integers u, v can be found such that $f^v = g^u$.* As before, it can be assumed that $u > 0$ and that u, $|v|$ are prime to each other.

Such functions are adequately described by means of the theorem:

[7.1] *If f, g are such that $f^v = g^u$, then there exists a function ϕ such that*

$$f = \phi^u, \qquad g = \phi^v. \tag{10}$$

In the case we are considering, when f, g are as described in **[5.2]**, ϕ must clearly be complete in $\langle a,b\rangle$ and with no interior node. Also, taking $f > I$ and $u > 0$, then $\phi > I$.

For proof, (10) can be verified at once by defining, as in § 6,

$$\phi = f^r g^s, \qquad ru+sv = 1. \tag{11}$$

COROLLARY. *Any finite number of functions related rationally in pairs in the same interval can be expressed as functional powers of one function.* This is easily proved by induction, using **[7.1]**.

* E. W. Hobson, *Theory of Functions of a Real Variable* (Cambridge, 1921), vol. i, chap. iv.

† Cf. Whitrow (**1**), 255, where this form is assumed.

8. Irrationally and semi-related functions

There remains to be considered the case when f^p, g^q do not intersect in (a, b) for any integers p, q except $p = q = 0$. As before, we take $f > I$.

We can prove in this case that, if $g < f$, then r can be found such that $g^r > f$. For, if $a < x_0 < b$, then r can be found such that $g^r(x_0)$ is as near b as we please and therefore greater than $f(x_0)$, whence $g^r > f$. Similarly, if $g > f$, then s can be found such that $f^s > g$. From these and the elementary properties of powers of f and g we deduce that an irrational number λ can be defined, as a section of the rationals, by

$$q\lambda \gtrless p \quad \text{according as} \quad g^q \gtrless f^p. \tag{12}$$

Writing $\lambda = \mathscr{L}_f(g)$, a similar definition at once gives

$$\mathscr{L}_f(f^p g^q) = p + q\lambda.$$

Since λ is irrational, the set of numbers $p+q\lambda$ for variable integers p, q is everywhere dense; it includes λ and all the integers. Symbols μ, ν will represent numbers of this set exclusively.

We can conveniently write f^μ for the function of the set $[f, g]$ which is such that

$$\mathscr{L}_f(f^\mu) = \mu.$$

It can easily be verified that this notation is consistent when μ is integral, and that for any μ, ν, and integer p,

$$f^\mu f^\nu = f^{\mu+\nu}, \qquad (f^\mu)^p = f^{p\mu},$$
$$f^\mu \gtrless I \text{ if } \mu \gtrless 0, \qquad f^\mu > f^\nu \text{ if } \mu > \nu. \tag{13}$$

DEFINITION. The *limit function* $L(x)$ of the set $[f, g]$ is defined by

$$L(x) = \lim_{\mu > 0} f^\mu(x), \tag{14}$$

by which is meant the limit as $\mu \to 0$ through positive values taken from the set $p+q\lambda$. From (13) we see that L exists in (a, b) (though it must not be assumed continuous), and that

$$x \leqslant L(x) \leqslant L(x') \quad (a < x < x' < b). \tag{15}$$

Also, by Theorem [**4.2**], for any μ,

$$Lf^\mu = f^\mu L. \tag{16}$$

As an example, consider (4) with $\lambda = (\log \alpha)/(\log \beta)$ irrational. Then $\mathscr{L}_f(g) = \lambda$, and in our notation, $f^\mu = \psi^{-1}\alpha^\mu\psi$, $\mu = p+q\lambda$. In this case, therefore, $L(x) \equiv x$.

It can be shown (see Part II) that not all pairs f, g have $L(x) \equiv x$, and the classification of commutative pairs is completed as follows, f, g being as described at the beginning of this section.

DEFINITIONS. *If $L(x) \equiv x$ in (a, b), then f, g are said to be* related irrationally. *If, however, there is an x_0 such that $L(x_0) \neq x_0$, then f, g are* semi-related.

We can now see that any two functions restricted as in **[5.2]** are either related, semi-related, or unrelated, and, if related, then either rationally or irrationally. These four classes are mutually exclusive, and *all pairs of functions taken from a set $[f, g]$ belong to the same class.*

It is worth noting that the class to which given functions belong may not be immediately obvious. If f, g are commutative, complete in the same interval, and without interior nodes and intersections, it is still possible for them to belong to any one of the four classes.

In the remainder of Part I we shall be largely concerned with related functions and those properties which lead to certain fundamental existence theorems.

9. Irrationally related functions

In this section I give some useful properties of irrationally related functions.

[9.1] *Two functions satisfying the conditions of* § 8 *are related irrationally if $L(x) = x$ throughout some interval, however small.*

To prove this, suppose that $L(x) = x$ $(a < c \leqslant x \leqslant d < b)$. Then, since $L(c) = c$, we see from (14) that there is a $\mu > 0$ such that $c < f^{\mu}(c) < d$. Writing $h = f^{\mu}$, we have

$$L(\xi) = \xi \quad (c \leqslant \xi \leqslant h(c)),$$

and from (16),

$$Lh^{p}(\xi) = h^{p}L(\xi) = h^{p}(\xi) \quad (c \leqslant \xi \leqslant h(c)).$$

Since h is complete in $\langle a, b\rangle$ and has no interior node, each point x in (a, b) can be expressed in the form $h^{p}(\xi)$ for some p and ξ $(c \leqslant \xi < h(c))$, as in (1). Hence $L(x) = x$ throughout (a, b), and f, g are irrationally related.

[9.2] *If f, g are irrationally related in $\langle a, b\rangle$ and $a < x_0 < b$, then the set $f^{\mu}(x_0)$ for variable μ is dense in (a, b).*

In this case, $L = I$, and the functions f^{μ} therefore converge uniformly to I as $\mu \searrow 0$. Hence, for any positive ϵ there is a positive ν such that

$$|f^{\nu}(x) - x| < \epsilon \quad (a \leqslant x \leqslant b).$$

Writing $h = f^\nu$, then $a < h^p(x_0) < b$ for any p, and from the above inequality with $x = h^p(x_0)$, we have

$$|h^{p+1}(x_0) - h^p(x_0)| < \epsilon.$$

The theorem now follows from the fact that (a, b) is covered by these intervals as p varies, since h^p belongs to the set f^μ.

The converse of this theorem is as follows:

[9.3] *If there exists x_0 such that the set $f^p g^q(x_0)$ is dense in (a, b), f, g being as described in* **[5.2]**, *then f, g are irrationally related.*

To prove this, we first see that the denseness of the set $f^p g^q(x_0)$ shows that f, g cannot be either unrelated (§ 6) or rationally related (§ 7). A limit function therefore exists, and the notation of § 8 can be used. Since $f^\mu(x_0)$ is dense in (a, b), it is dense at x_0, and $L(x_0) = x_0$ from (14). Hence,

$$Lf^\mu(x_0) = f^\mu L(x_0) = f^\mu(x_0),$$

and $L(x) = x$ at each point of $f^\mu(x_0)$, i.e. on a set dense in (a, b). It follows from (15) that $L(x) = x$ throughout (a, b), which proves the theorem.

It will be shown later (Part II) that in **[9.3]** *it is sufficient for the set to be dense in some interval, however small.*

10. A theorem on related functions

I now prove the following theorem:

[10.1] *Functions f, g, complete and commutative in $\langle a, b\rangle$, are related if c.m.i. functions h_n exist such that, in (a, b),*

(i) *h_n is commutative with f and g for all n;*

(ii) *$h_n(x) \to x$ as $n \to \infty$.*

It is important to note here, and again later, that by $x_n \to x$ we mean that every neighbourhood of x contains an x_n *other than x itself.* Thus, from (ii), no point can be a node of h_n for all large n.

The significance of the above theorem is its generality. It is not assumed as in **[5.2]** that f or g has no interior node, and it is not assumed that any two h's are commutative, or that the sequence h_n possesses an infinitesimal member, i.e. that there is a sequence of numbers ϵ_n such that

$$\lim_{n\to\infty} \frac{h_n(x) - x}{\epsilon_n}$$

exists and is non-zero for all x.

To prove the theorem we first show that, if F is any function of

the set $[f, g]$, then F has no node in (a, b). Let a', b' be consecutive nodes of F and choose x_0 in (a', b'). Then, from (ii), $a' < h_n(x_0) < b'$ for sufficiently large n, and, as in § 3, $F^p(x_0)$ and $F^p h_n(x_0)$ tend together to a' or b' as $p \to \pm\infty$. It follows from (i) that

$$h_n F^p(x_0) = F^p h_n(x_0),$$

and, letting $p \to \pm\infty$, we see that a' and b' are nodes of h_n. If now a' or b' is in (a, b), then this point is a node of h_n for all large n, and (ii) is contradicted. Thus F has no node in (a, b).

It follows that f, g belong to one of the four classes in $\langle a, b\rangle$, and that they cannot be unrelated. They are therefore either related or semi-related, and to rule out the latter possibility we need the following lemma, proved in Part II and quoted here without proof.

LEMMA. *If f, g are semi-related in $\langle a, b\rangle$, then there is an interval $\langle y, z\rangle$ within (a, b) such that $L(x) = z$ for $y \leqslant x \leqslant z$.*

Continuing the proof of [**10.1**], suppose that f, g are semi-related, y, z being defined as in the lemma, and choose x_0 in (y, z). Then since $h_n(x_0) \to x_0$, there is an m such that

$$y < h_n(x_0) < z \quad (n \geqslant m).$$

Now from the above lemma and Theorem [**4.2**] we have

$$h_n(z) = h_n L(x_0) = L h_n(x_0) = z \quad (n \geqslant m),$$

showing that z is a node of h_n for all sufficiently large n. This contradicts (ii), and f, g cannot therefore be semi-related. The only remaining possibility is that f, g are related, and the theorem is proved.

11. Related functions : canonical form

We are now in a position to prove a fundamental theorem:

[**11.1**] *If f, g are related in $\langle a, b\rangle$, then there exist constants α, β and a c.m.i. function ψ, where $\psi(a) = 0$, $\psi(b) = \infty$, such that*

$$f = \psi^{-1}\alpha\psi, \qquad g = \psi^{-1}\beta\psi. \tag{17}$$

The case of rationally related functions is soon dismissed, for the required forms follow at once when ϕ in (10) is expressed in canonical form by Theorem [**3.1**].

To prove the theorem when f, g are related irrationally, there is clearly no loss of generality in assuming $f > I$. Choose $\alpha > 1$ and write

$$\lambda = \mathscr{L}_f(g), \qquad \beta = \alpha^\lambda > 0. \tag{18}$$

Now choose x_0 in (a, b) and define a function ψ by

$$\psi f^{\mu}(x_0) = \alpha^{\mu}, \qquad \mu = p + q\lambda. \tag{19}$$

By Theorem [9.2], the set $f^{\mu}(x_0)$ is dense in (a, b). Also the set α^{μ} for variable μ is dense in $(0, \infty)$, and clearly these two sets are ordinally similar, with α^{μ} corresponding to $f^{\mu}(x_0)$. Hence ψ is defined and is c.m.i. at the points $f^{\mu}(x_0)$, and is therefore defined uniquely to be c.m.i. throughout $\langle a, b\rangle$ by continuation. The limits $\mu \to \pm\infty$ give ψ at a and b as stated.

We can now prove that (17) holds with this ψ and with β given by (18). First consider points $f^{\mu}(x_0)$. We have $g = f^{\lambda}$, and from (19),

$$\psi f f^{\mu}(x_0) = \psi f^{\mu+1}(x_0) = \alpha^{\mu+1} = \alpha\psi f^{\mu}(x_0),$$

$$\psi g f^{\mu}(x_0) = \psi f^{\mu+\lambda}(x_0) = \alpha^{\mu+\lambda} = \beta\psi f^{\mu}(x_0).$$

Hence $\psi f = \alpha\psi$ and $\psi g = \beta\psi$ at points of a dense set, and therefore throughout $\langle a, b\rangle$ since f, g, ψ are continuous. Relations (17) follow.

It is clear that ψ depends on α, as in § 3, and that, for a given α, ψ is determined except that it can be replaced by $\gamma\psi$ where γ is any positive constant.

12. Related sets

Theorem [11.1] gives sufficient conditions under which two functions can be expressed in canonical form simultaneously, i.e. with the same ψ. These conditions are also necessary, as can be seen from the example following (16). It is natural now to inquire under what conditions any number of functions can be expressed in canonical form simultaneously. I cannot give a complete answer to this question, but there are a number of theorems covering certain important sets of functions.

DEFINITION. *A number of functions constitute a* related set *if they can be expressed simultaneously in the canonical form described in* [3.1].

A necessary condition is that the functions must be related in pairs in the same interval, but this, we shall see, is not sufficient. The position when two functions of a set are related irrationally is simplified by the following theorem.

[12.1] *Let $f = \psi^{-1}\alpha\psi$ and $g = \psi^{-1}\beta\psi$ be irrationally related in $\langle a, b\rangle$. Then, if h is complete in $\langle a, b\rangle$ and commutative with f and g, we can express h in the form $\psi^{-1}\gamma\psi$ for some constant γ.*

To prove this, we can define $H = \psi h \psi^{-1}$, so that $h = \psi^{-1} H \psi$. From the commutative relations we get

$$H(\alpha x) = \alpha H(x), \qquad H(\beta x) = \beta H(x) \quad (x \geqslant 0).$$

Writing $H(x) = x\gamma(\log x)$, we find that the function γ is continuous and doubly periodic, of periods $\log\alpha$ and $\log\beta$. Since the ratio of these is not rational, γ must be constant, whence $H(x) \equiv \gamma x$ and $h = \psi^{-1}\gamma\psi$ as required.

An immediate consequence is the following:

COROLLARY. *A set of functions complete and commutative in pairs in the same interval is a related set if two of the functions are related irrationally.*

Another theorem on related sets is the following:

[12.2] *A finite set of functions related in pairs in the same interval is a related set.*

This follows from the above corollary if two of the functions are related irrationally. If all the relationships are rational, the theorem is a consequence of the corollary to **[7.1]**, the given functions being powers of one function which can be expressed in canonical form.

It is important now to realize that **[12.2]** cannot be extended to an infinite set of functions related in pairs. This can easily be illustrated by using the fact, discovered by Whitrow,* that, if $G > I$ is a function c.m.i. in an interval containing x_0, then for any k $(x_0 < k < G(x_0))$ there is a c.m.i. function F satisfying

$$F^2 = G, \qquad F(x_0) = k. \tag{20}$$

Let $h_0 > I$ be a function complete in $\langle a, b\rangle$, and choose $x_0\,(a < x_0 < b)$ and k, k_n so that

$$h_0(x_0) > k_1 > k_2 > \ldots > k > x_0.$$

Now apply (20) successively to find h_n, complete in $\langle a, b\rangle$, such that

$$h_n^2 = h_{n-1}, \qquad h_n(x_0) = k_n \quad (n \geqslant 1).$$

Then h_n are clearly related in pairs. Suppose they form a related set, so that $h_n = \psi^{-1}\alpha_n\psi$, where ψ is c.m.i. and $\alpha_0 > 1$. Then since $h_n^2 = h_{n-1}$, we find $\alpha_n = \alpha_0^{2^{-n}}$, so that $\alpha_n \to 1$. This implies that $h_n(x_0) \to x_0$, which is impossible since $h_n(x_0) > k > x_0$ for all n. The h_n do not, therefore, form a related set, despite the fact that they are related in pairs in the same interval.

* Loc. cit. (3), 270.

13. Infinite related sets

It was shown in the last section that not all infinite sets of related functions are related sets. I now discuss a wide class of infinite sets which are related, and prove two theorems, the first of which is contained in the second.

[**13.1**] *Functions f_n form a related set if they are related in pairs in the same interval and are such that $f_n \to I$.*

After the corollary to [**12.1**] it remains only to consider the case when f_n are related rationally in pairs in $\langle a, b\rangle$. By the corollary to [**7.1**], $f_1, f_2, \ldots, f_n$ are powers of a function ϕ_n, and ϕ_n, f_{n+1} are powers of ϕ_{n+1}. Also, ϕ_n is complete in $\langle a, b\rangle$, and $\phi_n \to I$, since f_n is a power of ϕ_n and $f_n \to I$. If now ϕ_n form a related set, then so do f_n. It is therefore sufficient for our theorem to prove that functions ϕ_n, complete in $\langle a, b\rangle$ and with no interior nodes, form a related set if ϕ_n is a power of ϕ_{n+1} and $\phi_n \to I$.

We can assume $\phi_1 > I$ without loss of generality. Suppose that ϕ_1, ϕ_n are related by $\phi_1 = \phi_n^{r_n}$, r_n thus being an integer which divides r_{n+1}. Choose $\alpha > 1$, and define $\alpha_n = \alpha^{1/r_n}$. Now choose x_0 in (a, b). Then it can be verified from the properties of ϕ_n that the sets $\phi_n^p(x_0)$, α_n^p are ordinally similar, corresponding members being given by the same n and p, and repetitions corresponding to repetitions. Also, the first set is dense in (a, b), as follows from $\phi_n \to I$ by the argument which proved Theorem [**9.2**], and the second set is dense in $(0, \infty)$ since $\alpha_n \to 1$. Hence a c.m.i. function ψ, with $\psi(a) = 0$, $\psi(b) = \infty$, is defined by

$$\psi\phi_n^p(x_0) = \alpha_n^p$$

at the points $\phi_n^p(x_0)$, and by continuation at all other points of $\langle a, b\rangle$. This function satisfies $\psi\phi_n = \alpha_n\psi$ at points of a dense set and therefore throughout $\langle a, b\rangle$ since all the functions involved are continuous. Hence $\phi_n = \psi^{-1}\alpha_n\psi$, which proves the theorem.

The following theorem is still more general. It is not now assumed *a priori* that the functions are related in pairs or that they have no interior nodes, nor is it assumed that the parameter of the set takes only values from a simple progression.

[**13.2**] *Let functions $f_t = f(x, t)$ exist for values of t belonging to an infinite set (t) having t_0 as a limit, t_0 being not necessarily finite or*

a member of (t). *Then the functions form a related set in* $\langle a, b\rangle$ *if they satisfy the following conditions, for all* t, t':

(i) f_t *is complete in* $\langle a, b\rangle$; (ii) $f_t f_{t'} = f_{t'} f_t$; (iii) $f_t \to I$ *as* $t \to t_0$.

To prove this, choose from (t) a sequence t_n such that $t_n \to t_0$ as $n \to \infty$. Then Theorem [**10.1**] can be applied to any two functions f_t, $f_{t'}$ with $f(x, t_n)$ in place of h_n, whence *any two functions of the set are related in* $\langle a, b\rangle$. If this relationship is irrational for one pair, the theorem follows at once from the corollary to [**12.1**]. There now remains only the case when all the functions are related rationally in pairs.

Applying Theorem [**13.1**] to the set $f(x, t_n)$ above, this set is related, and we can write $f_{t_n} = \psi^{-1}\alpha_n \psi$ where ψ has the usual properties. Now consider any remaining function f_t. A c.m.i. function F exists such that $F = \psi f_t \psi^{-1}$, i.e. $f_t = \psi^{-1} F \psi$, and the commutative relation between f_t, f_{t_n} gives

$$F(\alpha_n x) = \alpha_n F(x) \quad (x \geqslant 0).$$

Writing $F(x) = xG(\log x)$, then G is continuous and periodic of period $\log \alpha_n$. But, from (iii), $\alpha_n \to 1$, and it follows that G is constant, say $\alpha_{(l)}$. Hence $f_t = \psi^{-1}\alpha_{(l)}\psi$, and so for each t. This completes the proof.

The conditions laid down in the above theorem can be compared favourably with assumptions made by Whitrow* when proving that a certain set was what we now call 'related'. Whitrow assumed the existence of the commutative relation between all pairs of functions, of the derivative $\partial f(x,t)/\partial x$ at all points for all t, and of the 'infinitesimal coefficient' $\phi(x) = \{\partial f(x,t)/\partial t\}_{t=t_0}$, and assumed that $\int_c^x \frac{dx}{\phi(x)}$ is continuous and monotone. Whitrow gave an elegant *ad hoc* proof that a set of functions under the above conditions is related, and M'Crea† later proved the same by means of group properties. It has now been shown that the question of whether a derivative exists need not arise, and that it need not be assumed that the set possesses an infinitesimal member, or that the set constitutes a group.

Conclusion

This completes our investigation of related functions and related sets. In Part II semi-related functions will be considered in detail.

* Loc. cit. (**1**), 253. † Loc. cit., **25**.

Meanwhile it may be of interest to note that the present paper gives rise to many problems at present unsolved. For example, having classified all pairs of c.m.i. commutative functions, there now remains the problem of classifying three or more such functions. It should not be difficult to do this by investigating all functions commutative with the various pairs already considered.

Other problems, some of which arise in relativity, are of a more special nature. For example, if f, g are given commutative functions, are there functions F, G such that

$$F^2 = f, \qquad G^2 = g, \qquad FG = GF?$$

A fact which should be noted is that we have nowhere assumed differentiability. It is a simple matter, however, to prove that, if functions are given to be n-times differentiable, then the various functional equations here considered possess solutions also having this property. The existence of analytic solutions, given analytic functions, is by no means so clear, and there arise many questions which have not yet, as far as I know, been answered. For example, can analytic functions always be found commutative with a given analytic function? Again, can an analytic function satisfying the conditions of [**3.1**] be expressed in canonical form $\psi^{-1}\alpha\psi$ where ψ is now required to be analytic? If these questions have affirmative answers and if in [**11.3**] f, g are given analytic, is ψ also analytic?

Another possible generalization of the present work is the study of c.m.i. commutative functions of a variable element belonging to a compact closed *non-linear* ordered set. Many of the present ideas still apply, and it is probable that analogous results will be found to exist.

COMMUTATIVE FUNCTIONS (II)

By A. G. WALKER (*Liverpool*)

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14. $L(x)$ for semi-related functions

THE definitive property of semi-related functions is that $L(x)$ defined in § 8 is not identically x. I shall now prove the following theorem, which describes $L(x)$ for functions f, g semi-related in $\langle a,b\rangle$, the existence of such functions being assumed for the present and proved later (§ 16).

[14.1] *There is in (a,b) a set of intervals $\langle y,z\rangle$ which do not overlap or abut, and are such that the complementary set of points T in $\langle a,b\rangle$ contains no interval. In each interval $\langle y,z\rangle$, L is constant, being given by*

$$L(x) = z \quad (y \leqslant x \leqslant z), \tag{21}$$

and, at each point t of T,

$$L(t) = t. \tag{22}$$

Writing Y, Z for the point sets (y), (z) respectively, it follows that $L(x) = x$ *at each point of $Z+T$ and at no other point.*

We first prove that

$$L^2 = L. \tag{23}$$

Given $\mu > 0$, there is a ν such that $0 < 2\nu < \mu$ (μ, ν being numbers of the set $p+q\lambda$ as explained in § 8). Then $f^\nu > I$, $f^\nu > L$, and $f^\mu > f^{2\nu}$, and we have, using (16),

$$L \leqslant Lf^\nu = f^\nu L < f^{2\nu} < f^\mu.$$

Now let $\mu \searrow 0$. Then $\nu \searrow 0$, $f^\mu \to L$, and $f^\nu L \to L^2$, whence $L^2 = L$.

Let x_0 be a point at which $L(x_0) \neq x_0$, and write

$$z = L(x_0), \qquad y = \lim_{\mu>0} f^{-\mu}(z). \tag{24}$$

Then, from (23),

$$L(z) = L^2(x_0) = L(x_0) = z. \tag{25}$$

Since $f^{-\mu}$ increases as μ decreases,

$$\mu > 0 \supset y > f^{-\mu}(z) \supset f^\mu(y) > z,$$

and proceeding to the limit $\mu \searrow 0$, we get $L(y) \geqslant z$. Since $f^{-\mu} < I$ when $\mu > 0$, (24) gives $y \leqslant z$. Thus we have, from (15) and (25),

$$L(y) = z. \tag{26}$$

Consider any $\eta < y$. Then from the definition of y, there is a $\mu > 0$ such that $\eta < f^{-\mu}(z)$, and, since $\mu > 0$, we have

$$L(\eta) < f^\mu(\eta) < z. \tag{27}$$

Summing up, we have seen that

$$L(x_0) = L(y) = L(z) = z,$$

and, from (27) and (15),

$$L(\eta) < z, \qquad L(\zeta) > z \quad (\eta < y \leqslant z < \zeta).$$

Hence, finally, since L is non-decreasing, $L(x) = z$ in an interval containing x_0, the extent of the interval being $\langle y, z\rangle$ given by (24).

It is clear that no two such intervals can overlap, nor can they abut, being closed. The point a cannot belong to Y, nor b to Z, since the functions f^μ are complete in $\langle a, b\rangle$. There is thus a residual set of points, T, in $\langle a, b\rangle$. At each point t of T, (22) holds, for otherwise t would belong to one of the intervals $\langle y, z\rangle$. It follows from this and Theorem [**9.1**] that T contains no interval. Theorem [**14.1**] is thus fully established.

It can be shown from the description of Y, Z, T in [**14.1**] that T is nowhere dense but has the cardinal number of the continuum. Also, y is a limit-point of Y and of Z on the left, z is a limit-point on the right, and t is a limit-point on both sides.

An example of such sets is provided by the Cantor ternary set. Taking $a = 0$, $b = 1$, then the intervals $\langle y, z\rangle$ are $\langle \frac{1}{3}, \frac{2}{3}\rangle$, $\langle \frac{1}{9}, \frac{2}{9}\rangle$, $\langle \frac{7}{9}, \frac{8}{9}\rangle$, $\langle \frac{1}{27}, \frac{2}{27}\rangle$,..., i.e. the middle thirds of all intervals remaining in $(0, 1)$ at each stage. In terms of ternary numbers $0{\cdot}a_1a_2a_3\ldots$ ($a_n = 0, 1, 2$), the points of Z are the terminating numbers (other than 0) in which no digit is 1, and the points of Y are the terminating numbers in which the last, but no other, digit is 1. It follows that the points of T are the non-terminating numbers in which 1 does not occur, e.g. $0{\cdot}020202\ldots$ $(= \frac{1}{4})$.

15. The second limit-function

Another step-function arising out of semi-related functions is

$$M(x) = \lim_{\mu > 0} f^{-\mu}(x). \tag{28}$$

By arguments similar to those for L, it is found that M is monotone non-decreasing and satisfies

$$M(x) \leqslant x, \qquad Mf^\mu = f^\mu M, \qquad M^2 = M. \tag{29}$$

From (24) we have $M(z) = y$. But y is a limit-point of Y on the left, z is a limit-point of Z on the right, and M is non-decreasing. Hence, *for the same sets* Y, Z, T *as in* [**14.1**],

$$M(t) = t, \qquad M(x) = y \quad (y \leqslant x \leqslant z). \tag{30}$$

Thus M is a discontinuous step-function *similar* to L, i.e. is monotonic non-decreasing, and constant in each interval in which L is constant.

We see that L, M differ in that they are discontinuous at points of Y, Z respectively. They are not commutative, for it can be seen that $LM = L$, $ML = M$.

16. Existence of semi-related functions

We now construct functions f, g, complete in $\langle a, b\rangle$ and with no interior nodes, which will be shown to be semi-related.

Choose a set of intervals $\langle y, z\rangle$ in (a, b) having the properties described in [**14.1**], and choose constants

$$\alpha > 1, \qquad \beta = \alpha^{\lambda} \quad (\lambda \text{ irrational}). \tag{31}$$

This set of intervals and the set of numbers $\mu = p+q\lambda$ have the same order type and can therefore be correlated so that to each μ there corresponds $\langle y_\mu, z_\mu\rangle$, where

$$y_\mu < z_\mu < y_\nu \quad (\mu < \nu). \tag{32}$$

To each μ' *not* of the set $p+q\lambda$ there corresponds a point $t_{\mu'}$ of T such that

$$\mu \to \mu' \supset y_\mu \to t_{\mu'}, \qquad z_\mu \to t_{\mu'}. \tag{33}$$

A continuous monotonic non-decreasing function χ is therefore defined in $\langle a, b\rangle$ by

$$\chi(x) = \alpha^{\mu} \qquad (y_\mu \leqslant x \leqslant z_\mu) \qquad \chi(t_{\mu'}) = \alpha^{\mu'}. \tag{34}$$

We see that χ is not strictly increasing, and that χ^{-1} exists uniquely only at points $\alpha^{\mu'}$. If, however, t is any point of T, then χ^{-1} exists at all points such as $\alpha^{\mu}\chi(t)$, i.e. $\chi^{-1}\alpha^{\mu}\chi$ exists at all points of T.

Let f be constructed as follows. At points of T, Y, Z,

$$f(t) = \chi^{-1}\alpha\chi(t), \tag{35}$$

$$f(y_\mu) = y_{\mu+1}, \qquad f(z_\mu) = z_{\mu+1}, \tag{36}$$

and in each interval (y_μ, z_μ), f is chosen so that it is c.m.i. in $\langle y_\mu, z_\mu\rangle$.

It can be verified from the definitions of χ and f that f is c.m.i. in $\langle a, b\rangle$. Also, from (35), a and b but no other points of T are nodes of f. From (36) no point of $\langle y_\mu, z_\mu\rangle$ can be a node, the range of f in this interval being $\langle y_{\mu+1}, z_{\mu+1}\rangle$ which is outside the interval. *Hence f is complete in $\langle a, b\rangle$ and has no interior node.*

A function g, complete in $\langle a, b\rangle$ and commutative with f, can now

be constructed as shown in § 5. We must first consider an interval such as $\langle y_0, y_1\rangle$, y_1 being $f(y_0)$. At points of T, Y, Z in this interval, let

$$g(t) = \chi^{-1}\beta\chi(t), \tag{37}$$

$$g(y_\mu) = y_{\mu+\lambda}, \qquad g(z_\mu) = z_{\mu+\lambda}, \tag{38}$$

and in each interval (y_μ, z_μ) $(0 \leqslant \mu < 1)$ choose g so that it is c.m.i. in $\langle y_\mu, z_\mu\rangle$.

It can be verified that g is c.m.i. in $\langle y_0, y_1\rangle$. Also $g(y_1) = fg(y_0)$. Hence, as in § 5, g can be defined throughout (a, b) by

$$gf^p(\xi) = f^p g(\xi) \quad (y_0 \leqslant \xi < y_1;\ p = \pm 1, \pm 2, \ldots) \tag{39}$$

and is complete in $\langle a, b\rangle$ and commutative with f.

It will be shown that (37) now holds for *all* points of T. For, if $y_p < t < y_{p+1}$, then, from (35), $f^{-p}(t)$ is a point $\bar{t}$ of T in (y_0, y_1), and from (39), (35), and (37) we have, as required,

$$g(t) = gf^p(\bar{t}) = f^p g(\bar{t}) = \chi^{-1}\alpha^p\beta\chi(\bar{t}) = \chi^{-1}\beta\chi f^p(\bar{t}) = \chi^{-1}\beta\chi(t).$$

Also (38) now hold for *all* μ. For, if $\mu = p+\nu$ $(0 \leqslant \nu < 1)$ and p is an integer, then, from (36), $y_\mu = f^p(y_\nu)$, and, from (38) and (36),

$$g(y_\mu) = gf^p(y_\nu) = f^p g(y_\nu) = f^p(y_{\nu+\lambda}) = y_{p+\nu+\lambda} = y_{\mu+\lambda}.$$

Similarly for $g(z_\mu)$.

We are now in a position to prove that f, g are semi-related; we shall first prove that they are not unrelated or related rationally. From (35) and (37), f^p, g^q (p, q not both zero) cannot intersect at a point of T other than a, b, since $\alpha^p \neq \beta^q$. In $\langle y_\mu, z_\mu\rangle$, the range of values of f^p is $\langle y_{\mu+p}, z_{\mu+p}\rangle$ from (36), and the range of values of g^q is $\langle y_{\mu+q\lambda}, z_{\mu+q\lambda}\rangle$ from (38). From (33) and the fact that λ is irrational, we see that these two ranges have no value in common. Hence, f^p, g^q cannot intersect in $\langle y_\mu, z_\mu\rangle$, and so finally in (a, b).

The remaining possibilities are that f, g are either irrationally related or semi-related in $\langle a, b\rangle$, and we can use the notation of § 8. If $\nu = p+q\lambda$, then, from (36) and (38),

$$f^\nu(y_\mu) = f^p g^q(y_\mu) = f^p(y_{\mu+q\lambda}) = y_{\mu+p+q\lambda} = y_{\mu+\nu},$$

and similarly $$f^\nu(z_\mu) = z_{\mu+\nu}.$$

Consider now any μ and x $(y_\mu < x < z_\mu)$. Then, since f^ν is c.m.i.,

$$y_{\mu+\nu} = f^\nu(y_\mu) < f^\nu(x) < f^\nu(z_\mu) = z_{\mu+\nu}.$$

Taking $\nu > 0$, then $z_\mu < y_{\mu+\nu}$ from (32), and we have

$$z_\mu < f^\nu(x) < z_{\mu+\nu} \quad (\nu > 0).$$

Hence, in the limit $\nu \searrow 0$,

$$L(x) = z_\mu \neq x,$$

and f, g are therefore semi-related.

It can similarly be proved that $M(x) = y$, and it follows from the properties of L, M that the intervals $\langle y, z\rangle$ used to construct f, g are the intervals associated with f, g in Theorem [**14.1**].

17. Limit functions L_τ, M_τ

Associated with semi-related functions f, g are infinitely many functions L_τ, M_τ similar to L, M. These and the relations between them are interesting in themselves, and they help to throw light on the set $[f, g]$. In what follows, numbers ρ, σ, τ can take all values, and μ, ν are restricted as before to the set $p+q\lambda$.

Monotonic non-decreasing functions L_τ, M_τ can be defined by

$$L_\tau(x) = \lim_{\mu>\tau} f^\mu(x), \qquad M_\tau(x) = \lim_{\mu<\tau} f^\mu(x). \tag{40}$$

Thus $L_0 = L$, $M_0 = M$. For any ν,

$$L_\nu = \lim_{\mu>\nu} f^\mu = \lim_{\mu>0} f^\mu f^\nu = Lf^\nu = f^\nu L, \tag{41}$$

and similarly,

$$M_\nu = Mf^\nu = f^\nu M. \tag{42}$$

It follows at once that L_ν, M_ν are discontinuous step-functions similar to L, their constant values in $\langle y, z\rangle$ being $f^\nu(z)$, $f^\nu(y)$ respectively, and their values at a point t being $f^\nu(t)$.

For any number τ, we have, from (40),

$$L_\tau = \lim_{\nu>\tau} \lim_{\mu>\nu} f^\mu = \lim_{\nu>\tau} L_\nu,$$

and similarly,

$$M_\tau = \lim_{\nu>\tau} M_\nu.$$

Thus L_τ, M_τ are clearly constant in each interval $\langle y, z\rangle$, and so are similar to L.

18. Relations between limit-functions

I now prove the following identities relating the various limit-functions.

(i) $\quad L_\tau(x) > L_\sigma(x) \geqslant M_\sigma(x) > M_\rho(x) \quad (\tau > \sigma > \rho)$;

(ii) $\quad f^\mu L_\tau = L_\tau f^\mu = L_{\mu+\tau}, \qquad f^\mu M_\tau = M_\tau f^\mu = M_{\mu+\tau}$; (43)

(iii) $\quad \lim_{\sigma<\tau} L_\sigma = M_\tau, \qquad \lim_{\sigma>\tau} M_\sigma = L_\tau$; (44)

(iv) $\quad L_\sigma L_\tau = L_\tau L_\sigma = L_{\sigma+\tau}, \qquad M_\sigma M_\tau = M_\tau M_\sigma = M_{\sigma+\tau}$; (45)

(v) $\quad L_\sigma M_\tau = L_{\sigma+\tau}, \qquad M_\sigma L_\tau = M_{\sigma+\tau}$. (46)

Since $L_0 = L$, $M_0 = M$, the above are seen to include several previous results as special cases.

The inequalities in (i) come at once from the definitions of L_τ, M_τ. The first equality in (ii) is given by Theorem [**4.2**], and to complete, we have

$$L_\tau f^\mu = \lim_{\nu > \tau} f^\nu f^\mu = \lim_{\nu > \mu+\tau} f^\nu = L_{\mu+\tau}.$$

Similarly for $M_\tau f^\mu$. For (iii) we require

$$M_\tau L = M_\tau, \qquad L_\tau M = L_\tau.$$

These follow from the facts that L, M, L_τ, M_τ are all constant in each interval $\langle y, z\rangle$, and $M(y) = y$, $L(z) = z$, $L(t) = M(t) = t$.

Now, from (41) and (40),

$$\lim_{\sigma < \tau} L_\sigma = \lim_{\mu < \tau} L_\mu = \lim_{\mu < \tau} f^\mu L = M_\tau L = M_\tau.$$

Similarly for the second part of (iii). For (iv) we have, from (40) and (43),

$$L_\sigma L_\tau = \lim_{\mu > \sigma} f^\mu L_\tau = \lim_{\mu > \sigma} L_{\mu+\tau} = L_{\sigma+\tau} = L_\tau L_\sigma.$$

Similarly for the second set of equations.

For (v) we have, from (40), (43), (44),

$$L_\sigma M_\tau = \lim_{\mu > \sigma} f^\mu M_\tau = \lim_{\mu > \sigma} M_{\mu+\tau} = L_{\sigma+\tau}.$$

Similarly for the second equation.

19. Cross-section of $[f, g]$

It is instructive to examine the sets $L_\tau(x_0)$, $M_\tau(x_0)$ for all τ and any particular x_0, i.e. the limit-points of the cross-section of $[f, g]$ at x_0. We shall write l_τ for $L_\tau(x_0)$ and m_τ for $M_\tau(x_0)$.

Equation (45) with $\sigma = 0$ gives $L(l_\tau) = l_\tau$, so that l_τ is a point of the set $Z+T$, by Theorem [**14.1**]. If l_τ is a point of Z, then, from (30), $M(l_\tau)$ is the corresponding point of Y. From (46) with $\sigma = 0$ we find $M(l_\tau) = m_\tau$. Hence

[**19.1**] *If $l_\tau = m_\tau$, then l_τ is a point of T. If $l_\tau \neq m_\tau$, then m_τ, l_τ are corresponding points of Y, Z respectively.*

Consider any number x in (a, b). Then a number τ is defined as a section of the numbers $\mu = p+q\lambda$ by

$$\mu \gtrless \tau \quad \text{according as} \quad f^\mu(x_0) \gtrless x.$$

Clearly, for this τ, $l_\tau \geqslant x \geqslant m_\tau$. From this and the above theorem can be deduced the following:

[**19.2**] *To each point t of T there corresponds some τ such that $l_\tau = m_\tau = t$, and to each interval $\langle y, z\rangle$ there corresponds some τ such that $m_\tau = y$, $l_\tau = z$.*

We thus have an ordinal mapping of the interval (a, b) on to the continuum (τ), with each point of T and each interval $\langle y, z\rangle$ corresponding to one point of the continuum. The correspondence which carries out this mapping varies, of course, with x_0. It is not clear whether L_τ, M_τ for a given τ are necessarily equal or necessarily distinct at some points.

20. Continuity of limit-functions

The functions L_τ, M_τ were shown in § 17 to be constant in each interval $\langle y, z\rangle$, but the question of continuity at points of Y, Z, T was not then discussed. I now prove that:

[**20.1**] *At each point x of $Y+T$,*

$$L_\tau(x-0) = M_\tau(x-0) = M_\tau(x), \tag{47}$$

and at each point x of $Z+T$,

$$M_\tau(x+0) = L_\tau(x+0) = L_\tau(x).$$

Thus, M_τ is continuous at each y, and continuous on the right at each t; L_τ is continuous at each z, and continuous on the left at each t. At a point x of $Y+T$, L_τ may be discontinuous on the right, and is so if $L_\tau(x) \neq M_\tau(x)$. Similarly, at a point x of $Z+T$, M_τ may be discontinuous on the left, and is so if $L_\tau(x) \neq M_\tau(x)$.

To prove the theorem, let x be a point of $Y+T$, i.e. let it satisfy $M(x) = x$. Then, if $\epsilon > 0$, there is a $\mu < 0$ such that

$$x-\epsilon < f^\mu(x) < x,$$

whence
$$L_\tau(x-\epsilon) \leqslant L_\tau f^\mu(x) = L_{\tau+\mu}(x)$$

from (43). If now $\epsilon \searrow 0$, then $\mu \nearrow 0$, and we get from (44),

$$L_\tau(x-0) \leqslant M_\tau(x). \tag{48}$$

Now choose $t < M_\tau(x)$, t being a point of T, and write $\xi = M_{-\tau}(t)$. Then, from (45),

$$M_\tau(\xi) = M_\tau M_{-\tau}(t) = M(t) = t < M_\tau(x), \tag{49}$$

whence $\xi < x$ since M_τ is non-decreasing. Since $M_\tau(x)$ is a point of $Y+T$ by [**19.1**], it is a limit-point of T on the left, and we can let $t \nearrow M_\tau(x)$ through points of T. We then get, from (49), $M_\tau(\xi) \nearrow M_\tau(x)$, whence

$$M_\tau(x-0) = M_\tau(x). \tag{50}$$

Since $L_\tau(x-0) \geqslant M_\tau(x-0)$, (48) and (50) now give (47). The second part of the theorem can be similarly proved.

21. Conclusion

This completes our discussion of semi-related functions, but there still remain many problems. For example, it is probable that any two semi-related functions can be expressed in the forms $\chi^{-1}\alpha\chi(t)$, $\chi^{-1}\beta\chi(t)$ at points of T, as in § 16. Such forms, if valid, would lead to alternative proofs of the properties given in §§ 19, 20. Again, is there a simple way of expressing two semi-related functions? In all other classes of commutative functions, there is in each case a particularly simple and useful form—see (8), (10), and (17). In this case, however, the forms $\chi^{-1}\alpha\chi(t)$, etc., are inadequate since they do not describe the whole functions. There are perhaps other canonical forms, possibly analogous to the special canonical forms for matrices.

APPENDIX

It will now be shown that there is no real loss of generality when c.m.i. functions are assumed to be complete in some interval.

Consider f, given to be c.m.i. in $\langle a, b\rangle$ but not complete. If the interval and range of values of f are finite, then clearly the interval can be extended and f continued over the extension so that f becomes complete in some new interval containing $\langle a, b\rangle$. This does not affect f in $\langle a, b\rangle$, and from properties in the new interval can be deduced properties in $\langle a, b\rangle$.

If the interval or values of f are not finite (e.g. e^x at $-\infty$, and $-x^{-1}$ at $-\infty$ and 0), an extension as above is not possible. In this case all the functions involved can be transformed in such a way that the interval and ranges become finite but the important functional relations are unaltered. The interval can then be extended as before, and from properties of the new and extended functions can be deduced properties of the original functions.

The *transform of f under ϕ* is f^*, where

$$f^* = \phi f \phi^{-1},$$

ϕ being c.m.i. If f is c.m.i. in $\langle a, b\rangle$, then f^* is c.m.i. in $\langle \phi(a), \phi(b)\rangle$, which we may call the transform of $\langle a, b\rangle$.

Transforming all functions under the same ϕ, we find at once

$$I^* = I, \qquad (f^p)^* = (f^*)^p, \qquad (fg)^* = f^*g^*,$$

etc. Also, if $f > g$, then $f^* > g^*$. It is thus seen that all the properties of commutative functions and the equations connected with them are unaltered by a transformation such as the above.

A transformation which is such that the new interval and ranges of values are always finite is given by

$$\phi(x) \equiv x(1+x^2)^{-\frac{1}{2}}, \qquad \phi^{-1}(x) \equiv x(1-x^2)^{-\frac{1}{2}}.$$

For it is at once seen that the transform of $(-\infty, \infty)$ is $(-1, 1)$, and that for any f, $|f^*| \leqslant 1$.

As an example, consider $f(x) \equiv -x^{-1}$ in $\langle -\infty, 0\rangle$. Neither endpoint is a node, and a transformation is necessary before extension. Under the above transformation we get $f^*(x) \equiv (1-x^2)^{\frac{1}{2}}$ in $\langle -1, 0\rangle$. This function can be continued so that it is complete in $\langle a, b\rangle$ provided that $a < -1$, $b > 1$.

Consider now commutative c.m.i. functions f, g, assuming only that each exists in an interval sufficiently large for $fg = gf$ to hold in some interval, say $\langle a_0, b_0\rangle$. We shall prove that *this interval can be extended, after transformation if necessary, to one in which both functions are complete and commutative.*

Assuming that a transformation has already been applied if necessary, consider the situation on the right. If b_0 is a node of both f and g, the position is already as desired, so suppose that $f(b_0) \neq b_0$. Since $fg = gf$ in $\langle a_0, b_0\rangle$, f must exist as far to the right as b_1, the greater of b_0 and $g(b_0)$. This point cannot be a node of f, for b_0 is not a node, and

$$f(b_0) \neq b_0 \supset gf(b_0) \neq g(b_0) \supset fg(b_0) \neq g(b_0),$$

so that $g(b_0)$ is not a node.

Now let B be any number greater than both b_1 and $f(b_1)$, and choose f in $(b_1, B\rangle$ so that it is c.m.i. and has a node at B but not in (b_1, B). There are three cases to be considered.

(i) If $f(b_0) \geqslant b_1$, then f has no node in (b_0, B), g must already exist as far right as $f(b_0)$, and g can be defined in $(f(b_0), B)$ by

$$gf^n(\xi) = f^n g(\xi), \qquad b_0 \leqslant \xi < f(b_0) \quad (n = 1, 2, \ldots).$$

It is easily seen, as in § 5, that g is c.m.i. and commutative with f in (b_0, B), and that, in the limit, $g(B) = B$, as required.

(ii) If $f(b_0) < b_1 = g(b_0)$, then the above argument and definitions can be repeated with f and g interchanged.

(iii) If $f(b_0) < b_1 = b_0$, then g must already exist as far to the right as b_0, and can be defined in (b_0, B) by

$$gf^{-n}(\xi) = f^{-n}g(\xi), \qquad f(b_0) < \xi \leqslant b_0 \quad (n = 1, 2, \ldots).$$

As before, g has the required properties in (b_0, B), and $g(B) = B$.

In all cases, therefore, the interval has been extended to the right and f, g defined so that they remain c.m.i. and commutative and have the new right end-point as a node. Extending similarly to the left, we finally get an interval $\langle A, B\rangle$ in which f, g are complete and commutative, and of which $\langle a_0, b_0\rangle$ is a sub-interval.

LATTICE POINTS IN A CIRCULAR QUADRILATERAL BOUNDED BY THE ARCS OF FOUR CIRCLES

By KATHLEEN OLLERENSHAW (*Oxford*)

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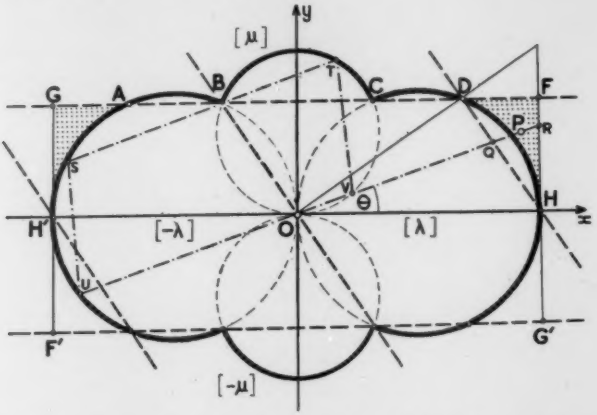
1. Introduction

RECENTLY Mordell* has proved some general theorems concerning lattice points in non-convex regions. In particular he has considered a circular quadrilateral bounded by arcs of equal circles which form a region resembling an asteroid. In this note I find similar results for another closed, non-convex circular quadrilateral $\mathscr{E}$, which is shown in the figure and presently defined. The proof of these results provides an extremely simple example of Mordell's method: probably as simple as any that can be found.



A lattice Λ of positive determinant Δ is defined by the equations

$$x = \alpha\xi+\beta\eta, \qquad y = \gamma\xi+\delta\eta, \qquad \Delta \equiv \alpha\delta-\beta\gamma > 0, \tag{1}$$

where α, β, γ, δ are real numbers and ξ, η run through all integers. Consider a certain homogeneous function $f(x, y)$ of degree one, so that $|f(x, y)| = |f(-x, -y)|$. The results are concerned with the best possible (i.e. the least) value of κ, independent of α, β, γ, δ, such that a point of Λ not O satisfies the inequality

$$|f(x, y)| \leqslant \kappa\sqrt{\Delta}. \tag{2}$$

* L. J. Mordell, *Proc. London Math. Soc.* (2) 48 (1945), 339–90.

If such a least value of κ (say k) exists, then necessarily a point not O of every lattice of positive determinant Δ satisfies

$$|f(x,y)| \leqslant k\sqrt{\Delta}, \tag{3}$$

and, further, there are lattices, referred to as *critical lattices*, for which the equality sign is required in (2).

The region $\mathscr{E}$ is bounded by the arcs of four circles $[\pm\lambda]$, $[\pm\mu]$, defined by the equations

$$x^2+y^2 = \pm\lambda x, \qquad x^2+y^2 = \pm\mu y.$$

More precisely, if P is any point of $\mathscr{E}$, then its coordinates (x,y) must satisfy the inequality

$$\min\{|(x^2+y^2)/\lambda x|,\ |(x^2+y^2)/\mu y|\} \leqslant 1, \tag{4}$$

where λ, μ are positive numbers. The transformation $x \to y$, $y \to x$ merely interchanges λ, μ in (4). There is therefore no loss of generality in taking $\lambda \geqslant \mu$. In the convenient terminology of figure-skating, the four defining circles form two 'eights' with common centre and perpendicular axes: in fact, the particular domain arose out of a demonstration of figure-skating on the frozen static-water tank of an Oxford college.

I prove the theorem:

THEOREM 1. *Every lattice Λ of determinant $\lambda^2\rho^2$, where*

$$\rho \equiv \sqrt{\{\lambda\mu/(\lambda^2+\mu^2)\}},$$

has a point not O inside $\mathscr{E}$, unless Λ is one of the critical lattices L, L' defined by the equations*

$$x = \lambda\xi+\mu\rho^2\eta, \qquad y = \lambda\rho^2\eta; \tag{5}$$

$$x = \lambda\xi-\mu\rho^2\eta, \qquad y = \lambda\rho^2\eta. \tag{6}$$

When $\lambda = \mu$, then $\rho^2 = \frac{1}{2}$ and the two lattices L, L' coincide.

It is clear from the homogeneity of the expression on the left of (4), that every lattice of determinant less than $\lambda^2\rho^2$ has a point not O inside $\mathscr{E}$. It follows that $k = 1/\lambda\rho$ is the best possible value of κ such that every lattice of positive determinant Δ has a point other than the origin which satisfies (2). This result can now be restated in full thus:

* For convenience I shall often call an *inner* point of a region a point *inside* the region: a point in a region is thus either a boundary point or a point inside the region.

THEOREM 2. *If λ, μ are any positive numbers ($\lambda \geqslant \mu$) and α, β, γ, δ any real numbers such that*

$$\Delta \equiv \alpha\delta - \beta\gamma > 0,$$

and if f_1, f_2 denote the functions

$$\begin{aligned} f_1 &\equiv (a\xi^2 + 2b\xi\eta + c\eta^2)/\lambda(\alpha\xi + \beta\eta), \\ f_2 &\equiv (a\xi^2 + 2b\xi\eta + c\eta^2)/\mu(\gamma\xi + \delta\eta), \end{aligned} \tag{7}$$

where $a \equiv \alpha^2+\gamma^2$, $b \equiv \alpha\beta+\gamma\delta$, $c \equiv \beta^2+\delta^2$, then integers ξ, η, not both zero, can be found for which

$$\min\{|f_1|, |f_2|\} \leqslant \sqrt{\Delta}/\lambda\rho, \tag{8}$$

where $$\rho \equiv \sqrt{\{\lambda\mu/(\lambda^2+\mu^2)\}}.$$

2. Proof of Theorem 1

2.1. The circle $[\mu]$ meets the circles $[\pm\lambda]$ in points B, C on the line $y = \lambda\rho^2$. In the figure this line meets the circles $[\pm\lambda]$ again in the points* A, D, so that

$$A \text{ is } (-\lambda^2\rho^2/\mu, \lambda\rho^2), \qquad B \text{ is } (-\mu\rho^2, \lambda\rho^2),$$
$$C \text{ is } (\mu\rho^2, \lambda\rho^2), \qquad D \text{ is } (\lambda^2\rho^2/\mu, \lambda\rho^2).$$

Further, H is the point $(\lambda, 0)$, F is $(\lambda, \lambda\rho^2)$, G is $(-\lambda, \lambda\rho^2)$; and H', F', G',... are the images of H, F, G,... in O. Thus

$$AC = \lambda^2\rho^2/\mu + \mu\rho^2 = \lambda = BD = OH;$$

further $OBDH$, $OACH$ are parallelograms and so

$$OB = HD, \qquad OA = HC.$$

2.2. We notice first that the lattice L generated by H, C and defined by (5) has no point other than the origin inside $\mathscr{E}$. For, since $AC = OH$, A is a point of L, and so L has no point inside the circles $[\pm\lambda]$. Moreover, $2\lambda\rho^2 \geqslant \mu$, since $\lambda^2 \geqslant \mu^2$, and so no point of the lines $y = \pm 2\lambda\rho^2$ lies inside the circles $[\pm\mu]$; hence L has no point inside $[\pm\mu]$, and therefore no point not O inside $\mathscr{E}$. Similarly, the lattice L' generated by H', B (or by H, B) has no point not O inside $\mathscr{E}$. By (5), (6), the lattices L, L' both have the determinant $\lambda^2\rho^2$.

2.3. Now the area of the rectangle $GFG'F'$ is $4\lambda^2\rho^2$, and so, by Minkowski's classical theorem on linear forms, every lattice of determinant $\lambda^2\rho^2$ has a point not O in $GFG'F'$. If this point lies *inside* $\mathscr{E}$, there is nothing to prove. It may therefore be taken as B or C,

* If $\lambda = \mu$, then, of course, A, B coincide and similarly C, D.

or it may be taken as lying in one or other of the triangles $\{DFH\}$, $\{AGH'\}$ with one curved side (stippled in the figure).

Let us first consider a lattice Λ of determinant $\lambda^2\rho^2$ having a point P in $\{DFH\}$. If OP meets the line DH in Q, then $OP \geqslant OQ$ and the area of the triangle OBQ, which is half the area of the parallelogram $OBDH$, is $\frac{1}{2}\lambda^2\rho^2$. Suppose that the lattice Λ is generated by P and a point W above OP, then the triangles WOP, OBQ both have the area $\frac{1}{2}\lambda^2\rho^2$. Hence, since $OP \geqslant OQ$, W is not further from OP than SBT, the line drawn parallel to OP through B to meet the circles $[-\lambda]$, $[\mu]$ in S, T; moreover, if W lies on SBT or SBT produced, then P coincides with Q, i.e. P is at H or D.

2.4. Write θ for the angle POH and R for the point of intersection of OP produced with HF or HF produced. Then, since D is the point $(\lambda^2\rho^2/\mu, \lambda\rho^2)$, OD is $y = (\mu/\lambda)x$, and hence

$$0 \leqslant \tan\theta \leqslant \mu/\lambda. \tag{9}$$

If, further, PO produced meets $[-\lambda]$ in U and OP meets $[\mu]$ in V, then the lines SU, TV (which are contained in $\mathscr{E}$) are parallel, since the quadrilaterals $BOUS$, $BOVT$ are cyclic. Hence

$$ST = UO + OV = \lambda\cos\theta + \mu\sin\theta,$$

i.e., by (9),

$$ST \geqslant \lambda(\cos\theta + \tan\theta\sin\theta) = \lambda\sec\theta = OR \geqslant OP, \tag{10}$$

the first equality sign holding only when $\theta = 0$ or $\tan^{-1}(\mu/\lambda)$. When $\theta = \tan^{-1}(\mu/\lambda)$, R lies outside $\{DFH\}$ and so cannot be P, and hence $ST = OP$ only when $\theta = 0$, i.e. when P, S, T are respectively at H, A, C.

Thus, if W does not lie on SBT or SBT produced, then the line through W parallel to OP intersects SU, TV in inner points of these lines and so also in inner points of $\mathscr{E}$, and thus it contains a segment lying entirely within $\mathscr{E}$ of length greater than ST ($\geqslant OP$); hence Λ has a point not O inside $\mathscr{E}$. Thus we may suppose W to be a point of SBT or SBT produced, so that P is at H or D as was shown at the end of § 2.3.

2.5. Suppose first that P is at H. If Λ has no point other than the origin inside $\mathscr{E}$, there are now two alternatives: either (i) A, C are lattice points, i.e. Λ is the lattice L; or (ii) B is a lattice point and so also D, since $BD = OH$, i.e. Λ is the lattice L'. Suppose next that P is at D. Then, by (10), ST is greater than OD, and some inner point of ST is a point of Λ. This lattice point is also

an inner point of $\mathscr{E}$ unless it coincides with B, i.e. unless Λ is the lattice L'. Thus every lattice other than L, L' of determinant $\lambda^2\rho^2$ which has a point in $\{DFH\}$ has a point not O inside $\mathscr{E}$. Similarly, every lattice other than L, L' of determinant $\lambda^2\rho^2$ which has a point in $\{AGH'\}$ has a point not O inside $\mathscr{E}$.

2.6. It remains to consider all other lattices of determinant $\lambda^2\rho^2$ containing either the point B or C. Suppose that Λ' is such a lattice generated by B and a point W' to the right of OB. Then the triangles OBW', OBQ both have the area $\frac{1}{2}\lambda^2\rho^2$, and so W' lies on HD or HD produced. Further, since $HD = OB$, Λ' must have a point on DH. But inner points of DH are also inner points of $\mathscr{E}$. Thus, either Λ' has a point not O inside $\mathscr{E}$, or D, H (and B) are lattice points, i.e. Λ' is the lattice L. Similarly, if C is a point of a lattice of determinant $\lambda^2\rho^2$, then either it has a point not O inside $\mathscr{E}$ or it is the lattice L'. This completes the proof of Theorem 1.

When $\lambda = \mu$, then $\rho^2 = \frac{1}{2}$, $\lambda^2\rho^2 = \frac{1}{2}\lambda^2$. Moreover, A, B then coincide and similarly C, D. Thus the lattices L, L' coincide—as is otherwise evident from equations (5), (6)—and there is just one critical lattice of $\mathscr{E}$. This result can be proved independently by a direct application of Minkowski's theorem, since the area of the square with vertices at $(\pm\lambda, 0)$, $(0, \pm\lambda)$ is $2\lambda^2$ and every point of this square except the vertices and mid-points of its sides is an inner point of $\mathscr{E}$.

I remark that, if in (1) we choose one of the two particular matrices

$$\begin{Bmatrix} \alpha & \beta \\ \gamma & \delta \end{Bmatrix} \equiv \begin{Bmatrix} \lambda & \pm\mu\rho^2 \\ 0 & \lambda\rho^2 \end{Bmatrix},$$

so that Λ is either L or L', then f_1, f_2 become

$$f_1' \equiv \{\lambda\xi^2 + \mu\rho^2(\eta^2 \pm 2\xi\eta)\}/(\lambda\xi \pm \mu\rho^2\eta),$$
$$f_2' \equiv \{\lambda\xi^2 + \mu\rho^2(\eta^2 \pm 2\xi\eta)\}/\mu\rho^2\eta,$$

where either the positive or the negative sign is to be taken throughout. These are the two 'critical' pairs of forms of f_1, f_2; that is to say, the forms for which the equality sign is required in (8).

3. Extensions of Theorem 1

It is easily seen that the region $\mathscr{E}$ can be extended in a variety of ways without altering Theorem 1. For, if $\mathscr{E}'$ is any simply-connected region which contains $\mathscr{E}$ and is such that no point other than the origin of the lattices L, L' is an inner point of $\mathscr{E}'$, then Theorem 1 clearly holds for the new region $\mathscr{E}'$.

In particular we may replace the arc BC of the circle $[\mu]$ by any arc (or arcs) through B, C (say α) such that any point (x, y) of α satisfies the inequalities

$$|x| \leqslant \mu\rho^2, \qquad \lambda\rho^2 \leqslant y \leqslant 2\lambda\rho^2, \qquad |(x^2+y^2)/\mu y| \geqslant 1.$$

We may then replace the arc $B'C'$ of $[-\mu]$ by a similar arc α' symmetrical with α. For example, the circles $[\pm\mu]$ may be replaced by

(i) the rectangle

$$|x| \leqslant \mu\rho^2, \qquad |y| \leqslant h \quad (\mu \leqslant h \leqslant 2\lambda\rho^2);$$

(ii) any pair of ellipses

$$\lambda(2b-\lambda\rho^2)x^2+\rho^2\mu^2(y^2 \mp 2by) = 0,$$

where

$$\tfrac{1}{2}\mu \leqslant b \leqslant \lambda\rho^2;$$

(iii) any ellipse

$$x^2\lambda^2\rho^4+y^2(a^2-\mu^2\rho^4) = a^2\lambda^2\rho^4,$$

where $\frac{4}{5}\mu^2\rho^4 \leqslant a^2 \leqslant \frac{1}{2}\lambda\mu\rho^2$, provided that $\mu/\lambda \leqslant \sqrt{\frac{8}{5}}$;

(iv) the lemniscate

$$(\lambda^2-\mu^2)(x^2+y^2)^2+\lambda^2\mu^2(x^2-y^2) = 0,$$

provided that $\mu/\lambda \leqslant \sqrt{(2\sqrt{3}-3)}$.

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BASIC FUNCTIONS AND POLYNOMIAL SEQUENCES

By F. H. JACKSON (*Eastbourne*)

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1. Introduction

THIS paper is an investigation of the properties of certain basic functions which give rise to sequences of polynomials. The polynomials are natural extensions of the well-known numerical additive sequences (u), (v) associated with the names of Fibonacci and the French mathematician E. Lucas (**6**). The polynomials possess properties which may be of some interest in the theory of numbers, more especially in the theory of prime numbers. I define two polynomials,

$$f_n(\lambda) \equiv (\omega_1^n - \omega_2^n)/(\omega_1 - \omega_2), \qquad F_{n+1}(\lambda) \equiv c_1 f_{n-1}(\lambda) + c_2 f_n(\lambda),$$

in which

$$\omega_1 = \tfrac{1}{2}\{1+\sqrt{(1+4\lambda)}\}, \qquad \omega_2 = \tfrac{1}{2}\{1-\sqrt{(1+4\lambda)}\}, \tag{1}$$

n is any integer positive or negative or zero, and c_1, c_2 are arbitrary constants. From the definition it follows that

$$f_{-n}(\lambda) = (-)^{n-1} f_n(\lambda).$$

The polynomial f_n is a natural analogue of u_n in the Fibonacci sequence (u): 0, 1, 1, 2, 3, 5,..., u_n,.... It satisfies a recurrence formula

$$f_{n+1}(\lambda) = \lambda f_{n-1}(\lambda) + f_n(\lambda).$$

Explicitly the sequence is

$$(f)\text{: } 0,\ 1,\ 1,\ 1+\lambda,\ 1+2\lambda,\ 1+3\lambda+\lambda^2,\ 1+4\lambda+3\lambda^2+\lambda^3,\ldots, f_n(\lambda),\ldots,$$

in which

$$f_n(\lambda) = \left\{1+\binom{n-2}{1}\lambda+\ldots+\binom{n-r-1}{r}\lambda^r+\ldots\right\}, \tag{2}$$

and the final term is $\lambda^{\frac{1}{2}(n-1)}$ or $\frac{1}{2}n\lambda^{\frac{1}{2}n-1}$ according as n is odd or even. It can be expressed either as a determinant or as a continued fraction. Macmahon (**7**) uses the series $1+\binom{n-2}{1}+\ldots+\binom{n-r-1}{r}+\ldots$ for the enumeration of certain *select partitions* derived from the

Rogers–Ramanujan series. Rado (8) has effected the resolution of $f_n(\lambda)$ into its irreducible factors in the rational field.

I define two basic functions which involve the polynomials F already defined in (1). They are

$$\left.\begin{aligned} S_r(c_1, c_2, c_3, c_4, \lambda, \mu, q) &\equiv F_{1+r}(\lambda)F_{1+r}(\mu)+ \\ &\quad + \sum_{n=1}^{\infty}\{F_{1+r+n}(\lambda)F_{1+r-n}(\mu)+F_{1+r+n}(\mu)F_{1+r-n}(\lambda)\}q^{n^2}, \\ T(c_1, c_2, c_3, c_4, \lambda, \mu, q) & \\ &\equiv \sum_{n=-\infty}^{+\infty}\{F_{2+r+n}(\lambda)F_{1+r-n}(\mu)+F_{2+r+n}(\mu)F_{1+r-n}(\lambda)\}q^{n(n-1)}. \end{aligned}\right\} \quad (3)$$

When $\mu = \lambda$, these functions take simpler forms, in which cases I shall use $S_r(\lambda)$, $T_r(\lambda)$ to denote briefly $S_r(c_1, c_2, \lambda, q)$, etc.

2. The associated q^θ-equations

In previous papers in this Journal (4) I have discussed certain basic differential equations in the notation of a differential symbolic operator

$$q^\theta \equiv \exp\left(x\frac{d}{dx}.\log q\right).$$

I here write four equations:

$$\{\lambda x^2q^{2\theta+1}+xq^\theta-1\}y_\lambda = 0, \quad (4)$$

$$\{\mu x^2q^{2\theta+1}+xq^\theta-1\}y_\mu = 0, \quad (5)$$

$$\{\lambda^3x^6q^{3\theta+6}-\lambda(1+\lambda)x^4q^{2\theta+2}-(1+\lambda)x^2q^\theta+1\}Y_\lambda = 0, \quad (6)$$

$$\{\lambda^2\mu^2x^8q^{4\theta+12}-\lambda\mu x^6q^{3\theta+6}-(\lambda+\mu+2\lambda\mu)x^4q^{2\theta+2}-x^2q^\theta+1\}Y_{\lambda\mu} = 0. \quad (7)$$

These equations are not independent; they form together one of the simplest examples of my general theorem (4):

If we have two homogeneous linear q^θ-equations of the second order, then we can form equations of the third order and fourth order with solutions that are products of the solutions of the two equations of the second order conditionally on the existence of certain invariants among the functional coefficients of the equations. Reference to my papers will show that the above equations satisfy the necessary conditions, so that the solution Y can be equated to y_λ^2, and more generally $Y_{\lambda\mu}$ equated to the product $y_\lambda y_\mu$. The basic theorem (4) *is a generalization of a well-known and useful theorem in linear differential equations due to Appell. It may be noted that the q^θ-equations are in fact linear differential equations of infinite order.*

3. Solutions of the q^θ-equations

In this section I merely state solutions of equations (4)–(7) in the form of basic Laurent series and, for convenience in reading, defer to the final section of the paper the method by which they are obtained. These solutions are

$$y_\lambda = \sum_{n=-\infty}^{+\infty} \{c_1 f_{n-1}(\lambda) + c_2 f_n(\lambda)\} x^n q^{\frac{1}{2}n(n-1)}, \tag{8}$$

where c_1, c_2 are arbitrary constants.

In terms of the F polynomials this is

$$\sum_{n=-\infty}^{+\infty} F_{n+1}(c_1, c_2, \lambda) x^n q^{\frac{1}{2}n(n-1)}; \tag{9}$$

y_μ is of the same form as y_λ, but with μ, c_3, c_4 replacing λ, c_1, c_2;

$$Y_\lambda = \sum_{n=-\infty}^{+\infty} \left\{ \begin{array}{l} (\rho x^{2n} q^{n(n-1)} + \rho' x^{2n+1} q^{n^2}) f_n(\lambda) f_{n-1}(\lambda) + \\ + (\sigma x^{2n} q^{n(n-1)} + \sigma' x^{2n+1} q^{n^2}) f_n(\lambda) f_{n-2}(\lambda) + \\ + (\tau x^{2n} q^{n(n-1)} + \tau' x^{2n+1} q^{n^2}) f_{n-1}(\lambda) f_{n-2}(\lambda) \end{array} \right\}, \tag{10}$$

$$Y_{\lambda\mu} = \sum_{n=-\infty}^{+\infty} \left\{ \begin{array}{l} (\rho x^{2n} q^{n(n-1)} + \rho' x^{2n+1} q^{n^2}) [f_n(\lambda) f_{n-1}(\mu) + f_n(\mu) f_{n-1}(\lambda)] + \\ + (\sigma x^{2n} q^{n(n-1)} + \sigma' x^{2n+1} q^{n^2}) [f_n(\lambda) f_{n-2}(\mu) + f_n(\mu) f_{n-2}(\lambda)] + \\ + (\tau x^{2n} q^{n(n-1)} + \tau' x^{2n+1} q^{n^2}) [f_{n-1}(\lambda) f_{n-2}(\mu) + f_{n-1}(\mu) f_{n-2}(\lambda)] \end{array} \right\}, \tag{11}$$

where ρ, σ, τ, etc. are arbitrary constants.

The solution $Y_{\lambda\mu}$ is symmetrical in λ, μ and reduces to the solution Y_λ when $\mu = \lambda$. This is what we should expect from inspection of the forms of the q^θ-equations. Since we can equate Y_λ and y_λ^2, and the series are Laurent series, we can, moreover, ensure also that Y_λ shall be identical with y_λ^2 by a suitable choice of the constants ρ, σ, τ, ρ', σ', τ', yet leaving c_1, c_2 arbitrary (y_λ^2 is an absolutely convergent series when $q < 1$). The coefficient of x^{2r} in

$$\left\{ \sum_{n=-\infty}^{+\infty} F_{n+1}(c_1, c_2, \lambda) x^n q^{\frac{1}{2}n(n-1)} \right\}^2$$

is

$$q^{r(r-1)} F_{1+r}^2(\lambda) + 2 \sum_{n=1}^{\infty} F_{1+r+n}(\lambda) F_{1+r-n}(\lambda) q^{n^2+r(r-1)} \equiv S_r(\lambda, c_1, c_2, q) q^{r(r-1)}. \tag{12}$$

We can equate (12) with the coefficient of x^{2r} in Y_λ, namely with

$$q^{r(r-1)} \{\rho f_r(\lambda) f_{r-1}(\lambda) + \sigma f_r(\lambda) f_{r-2}(\lambda) + \tau f_{r-1}(\lambda) f_{r-2}(\lambda)\}. \tag{13}$$

4. Determination of the arbitrary constants $\rho, \sigma, \tau, \ldots$ in $Y_\lambda = y_\lambda^2$

This is effected very simply by giving to r in (10) the values 0, 1, 2 in succession, while noting that $f_0(\lambda)$ is zero, and equating (10) to (12), whence

$$\left.\begin{aligned} \tau f_{-1}(\lambda) f_{-2}(\lambda) &= S_0(c_1, c_2, \lambda, q) \\ \sigma f_1(\lambda) f_{-1}(\lambda) &= S_1(c_1, c_2, \lambda, q) \\ \rho f_1(\lambda) f_2(\lambda) &= S_2(c_1, c_2, \lambda, q) \end{aligned}\right\}. \tag{14}$$

On substituting in (13) the values of ρ, σ, τ obtained from these equations, we find by equating (12) to (13) an interesting property of the S functions: namely

$$\begin{aligned} S_r(c_1, c_2, \lambda, q) = {} & \frac{f_r(\lambda) f_{r-1}(\lambda)}{f_1(\lambda) f_2(\lambda)} S_2(c_1, c_2, \lambda, q) + \\ & + \frac{f_r(\lambda) f_{r-2}(\lambda)}{f_1(\lambda) f_{-1}(\lambda)} S_1(c_1, c_2, \lambda, q) + \frac{f_{r-1}(\lambda) f_{r-2}(\lambda)}{f_{-1}(\lambda) f_{-2}(\lambda)} S_0(c_1, c_2, \lambda, q). \end{aligned} \tag{15}$$

By a similar procedure in the case of the coefficients x^{2r+1} in the equation $Y_\lambda = y_\lambda^2$ we find also a property of the T functions:

$$\begin{aligned} T_r(c_1, c_2, \lambda, q) = {} & \frac{f_r(\lambda) f_{r-1}(\lambda)}{f_1(\lambda) f_2(\lambda)} T_2(c_1, c_2, \lambda, q) + \\ & + \frac{f_r(\lambda) f_{r-2}(\lambda)}{f_1(\lambda) f_{-1}(\lambda)} T_1(c_1, c_2, \lambda, q) + \frac{f_{r-1}(\lambda) f_{r-2}(\lambda)}{f_{-1}(\lambda) f_{-2}(\lambda)} T_0(c_1, c_2, \lambda, q), \end{aligned} \tag{16}$$

where

$$T_r = \sum_{n=-\infty}^{+\infty} \{F_{2+r+n}(\lambda) F_{1+r-n}(\lambda) q^{n(n-1)}\}.$$

5. The constants $\rho, \sigma, \tau, \ldots$ in $Y_{\lambda\mu} = y_\lambda y_\mu$

In this case the method used for determining the values of the constants which will make the two series identical is precisely similar to that used in the simpler case of Y_λ. Reference to (11) will show that we need only replace the product pairs of the form $f_r(\lambda) f_s(\lambda)$ in (10), (12) by double pairs of the form $\{f_r(\lambda) f_s(\mu) + f_r(\mu) f_s(\lambda)\}$. Then

$$\left.\begin{aligned} & S_r(c_1, c_2, c_3, c_4, \lambda, \mu, q) \\ & = \frac{f_r(\lambda) f_{r-1}(\mu) + f_r(\mu) f_{r-1}(\lambda)}{f_1(\lambda) f_2(\mu) + f_1(\mu) f_2(\lambda)} S_2(c_1, c_2, c_3, c_4, \lambda, \mu, q) + \\ & \quad + \frac{f_r(\lambda) f_{r-2}(\mu) + f_r(\mu) f_{r-2}(\lambda)}{f_1(\lambda) f_{-1}(\mu) + f_1(\mu) f_{-1}(\lambda)} S_1(c_1, c_2, c_3, c_4, \lambda, \mu, q) + \\ & \quad + \frac{f_{r-1}(\lambda) f_{r-2}(\mu) + f_{r-1}(\mu) f_{r-2}(\lambda)}{f_{-1}(\lambda) f_{-2}(\mu) + f_{-1}(\mu) f_{-2}(\lambda)} S_0(c_1, c_2, c_3, c_4, \lambda, \mu, q) \end{aligned}\right\}, \tag{17}$$

with a similar formula for T_r, T_2, T_1, T_0, the coefficients in $f(\lambda)$, $f(\mu)$ remaining unchanged.

6. Certain identities

From (15), (16) we can now deduce results of some interest in the additive and other related sequences both polynomial and numerical.

If l, m, n be any integers positive or negative or zero, then

$$F_m F_n f_l f_{l-1} f_{-1} f_{-2} + F_{m-1} F_{n-1} f_l f_{l-2} f_2 f_{-2} + \\ + F_{m-2} F_{n-2} f_{l-1} f_{l-2} f_1 f_2 = F_{l+m-2} F_{l+n-2} f_1 f_2 f_{-1} f_{-2}. \quad (18)$$

This formula relating to products of six polynomials is, however, a simple case of much more general formulae which are obtained in later sections of the paper. To obtain (18) we equate coefficients of q^{n^2} in (15), and similarly coefficients of $q^{n(n-1)}$ in (16), thus obtaining the following two identities:

$$F_{3+n} F_{3-n} f_r f_{r-1} f_{-1} f_{-2} + F_{2+n} F_{2-n} f_r f_{r-2} f_2 f_{-2} + \\ + F_{1+n} F_{1-n} f_{r-1} f_{r-2} f_1 f_2 = F_{1+r+n} F_{1+r-n} f_1 f_2 f_{-1} f_{-2}, \quad (19)$$

$$F_{4+n} F_{3-n} f_r f_{r-1} f_{-1} f_{-2} + F_{3+n} F_{2-n} f_r f_{r-2} f_2 f_{-2} + \\ F_{2+n} F_{1-n} f_{r-1} f_{r-2} f_1 f_2 = F_{2+r+n} F_{1+r-n} f_1 f_2 f_{-1} f_{-2}. \quad (20)$$

Here (20) differs from (19) only in the leading F of each term having its rank increased by unity. Thus $4+n$ replaces $3+n$, and so on. We now make use of the recurrence formula $\lambda F_{m-1} + F_m = F_{m+1}$. On multiplying (19) throughout by λ and adding term by term to (20) we obtain a further increase of rank in the leading F polynomials of each term, namely

$$F_{5+n} F_{3-n} f_r f_{r-1} f_{-1} f_{-2} + \ldots = F_{3+r+n} F_{1+r-n} f_1 f_2 f_{-1} f_{-2}. \quad (21)$$

Repetition of this process $m-3$ times gives us, after a mere change of subscript letters, the formula (18) as stated above. For a simple numerical example write $\lambda = c_1 = c_2 = 1$. Then all the polynomials reduce to Fibonacci numbers and so, since $u_2 = -u_{-2}$ and $u_1 = u_{-1}$, we have

$$u_m u_n u_{l-1} u_l + u_{m-1} u_{n-1} u_{l-2} u_l - u_{m-2} u_{n-2} u_{l-2} u_{l-1} = u_{l+m-2} u_{l+n-2}, \quad (22)$$

which is easily verified. The formula (21), however, includes other

simple sequences when λ is given various integer values. Such, however, will *not be additive*. Thus, if $\lambda = 2$, we have a sequence

$$f(\lambda)\colon\ 1,\ 1,\ 3,\ 5,\ 11,\ 21,\ldots. \tag{23}$$

7. Identities for sequences with two parameters λ, μ

More general identities involving λ, μ, c_1, c_2, c_3, c_4 can now be found by equating coefficients of q^{n^2} in the formula (17) for

$$S_r(c_1, c_2, c_3, c_4, \lambda, \mu, q)$$

and by equating coefficients of $q^{n(n-1)}$ in the similar formula for T_r. From the coefficients of q^{n^2} in (17) we find

$$\begin{aligned}
&\{F_{1+r+n}(\lambda)F_{1+r-n}(\mu)+F_{1+r+n}(\mu)F_{1+r-n}(\lambda)\}\{f_1(\lambda)f_2(\mu)+f_1(\mu)f_2(\lambda)\}\times\\
&\quad\times\{f_1(\lambda)f_{-1}(\mu)+f_1(\mu)f_{-1}(\lambda)\}\{f_{-1}(\lambda)f_{-2}(\mu)+f_{-1}(\mu)f_{-2}(\lambda)\}\\
&\equiv\{F_{3+n}(\lambda)F_{3-n}(\mu)+F_{3+n}(\mu)F_{3-n}(\lambda)\}\{f_r(\lambda)f_{r-1}(\mu)+f_r(\mu)f_{r-1}(\lambda)\}\times\\
&\quad\times\{f_1(\lambda)f_{-1}(\mu)+f_1(\mu)f_{-1}(\lambda)\}\{f_{-1}(\lambda)f_{-2}(\mu)+f_{-1}(\mu)f_{-2}(\lambda)\}+\\
&+\{F_{2+n}(\lambda)F_{2-n}(\mu)+F_{2+n}(\lambda)F_{2-n}(\mu)\}\{f_r(\lambda)f_{r-2}(\mu)+f_r(\mu)f_{r-2}(\mu)\}\times\\
&\quad\times\{f_1(\lambda)f_2(\mu)+f_1(\mu)f_2(\lambda)\}\{f_{-1}(\lambda)f_{-2}(\mu)+f_{-1}(\mu)f_{-2}(\lambda)\}+\\
&+\{F_{1+n}(\lambda)F_{1-n}(\mu)+F_{1+n}(\mu)F_{1-n}(\lambda)\}\{f_{r-1}(\lambda)f_{r-2}(\mu)+f_{r-1}(\mu)f_{r-2}(\lambda)\}\times\\
&\quad\times\{f_1(\lambda)f_2(\mu)+f_1(\mu)f_2(\lambda)\}\{f_{-1}(\lambda)f_{-2}(\mu)+f_{-1}(\mu)f_{-2}(\lambda)\}.
\end{aligned}\tag{24}$$

From the coefficients of $q^{n(n-1)}$ we can find a second formula differing from the above only in that the rank (subscript) of the leading F in each product-pair will be increased by unity as in the simpler case of (19), (20). Thus in place of $F_{1+r+n}(\lambda)F_{1+r-n}(\mu)$ we shall have $F_{2+r+n}(\lambda)F_{1+r-n}(\mu)$, and so on.

The f coefficients remain unchanged. I need not give this second identity at length. We cannot, however, use these identities to obtain formulae with *successive* increases of rank in the leading F. It can be seen at once that multiplication either by λ or μ or $\lambda\mu$ throughout (24) followed by addition to the unwritten formula for T will *not* give us a symmetrical λ, μ formula of the type

$$F_{3+r+n}(\lambda)F_{1+r-n}(\mu)+F_{3+r+n}(\mu)F_{1+r-n}(\lambda),$$

but only an unsymmetrical formula in triple pairs. Yet a minor generalization can be found by making $\mu = \lambda$ while we retain all four arbitrary coefficients c_1, c_2, c_3, c_4. In this case multiplication by

λ throughout and a succession of additions give us a formula much more general than (18), namely

$$\{F_{l+m-2}(c_1,c_2,\lambda)F_{l+n-2}(c_3,c_4,\lambda)+ \\ +F_{l+m-2}(c_3,c_4,\lambda)F_{l+n-2}(c_1,c_2,\lambda)\}f_1f_2f_{-1}f_{-2} \\ =\{F_m(c_1,c_2,\lambda)F_n(c_3,c_4,\lambda)+F_m(c_3,c_4,\lambda)F_n(c_1,c_2,\lambda)\}f_lf_{l-1}f_{-1}f_{-2}+ \\ +\{F_{m-1}(c_1,c_2,\lambda)F_{n-1}(c_3,c_4,\lambda)+F_{m-1}(c_3,c_4,\lambda)F_{n-1}(c_1,c_2,\lambda)\}f_lf_{l-2}f_2f_{-2}+ \\ +\{F_{m-2}(c_1,c_2,\lambda)F_{n-2}(c_3,c_4,\lambda)+F_{m-2}(c_3,c_4,\lambda)F_{n-2}(c_1,c_2,\lambda)\}f_{l-1}f_{l-2}f_1f_2. \tag{25}$$

This can be expressed in an unlimited number of ways by varied numerical choice of c_1, c_2, c_3, c_4, λ. It contains a fivefold infinity of numerical sequences. Thus, if (f) is 0, 1, 1, 2, 3, 5,..., u_n,..., $F(c_1,c_2,\lambda)=1$, 3, 4, 7,..., $F(c_3,c_4,\lambda)=1$, 8, 9, 17,..., $l=3$, $m=4$, $n=6$, then each side of (25) reduces to 2489.

8. The most general forms

Finally, the *most* general results can be found by using q^θ-equations each with two parameters

$$\left.\begin{aligned}\{\lambda x^2q^{2\theta+1}+\mu xq^\theta-1\}y_{\lambda\mu}&=0\\ \{\lambda' x^2q^{2\theta+1}+\mu' xq^\theta-1\}y_{\lambda'\mu'}&=0\end{aligned}\right\}. \tag{26}$$

By the theorem of my paper (**4**) an equation of the fourth order can be derived from these, namely

$$\{\lambda^2\mu^2\lambda'^2\mu'^2x^8q^{4\theta+12}-\lambda\lambda'\mu\mu'x^6q^{3\theta+6}-(\lambda\mu'+\lambda'\mu+2\lambda\lambda')x^4q^{2\theta+2}- \\ -\mu\mu'x^2q^\theta+1\}Y_{\lambda\mu\lambda'\mu'}=0. \tag{27}$$

The solution of (26) differs from (8) only in that the polynomial $f(\lambda)$ is replaced by

$$f(\lambda,\mu)=\left\{\mu^n+\binom{n-2}{1}\lambda\mu^{n-2}+\ldots+\binom{n-r-1}{r}\lambda^r\mu^{n-2r}+\ldots\right\}, \tag{28}$$

i.e.

$$y_{\lambda\mu}=\sum_{-\infty}^{+\infty}\{c_1f_{n-1}(\lambda,\mu)+c_2f_n(\lambda,\mu)\}x^nq^{\frac{1}{2}n(n-1)} \\ \equiv\sum_{-\infty}^{+\infty}F_{n+1}(c_1,c_2,\lambda,\mu)x^nq^{\frac{1}{2}n(n-1)}.$$

In this case the recurrence-equation is

$$f_{n+1}(\lambda,\mu)=\lambda f_{n-1}(\lambda,\mu)+\mu f_n(\lambda,\mu).$$

The most general S_r contains four parameters and four arbitrary constants. It is

$$S_r(c_1, c_2, c_3, c_4, \lambda, \mu, \lambda', \mu', q) = F_{1+r}(\lambda, \mu)F_{1+r}(\lambda', \mu') + \\ + \sum_{n=1}^{\infty} \{F_{1+r+n}(\lambda, \mu)F_{1+r-n}(\lambda', \mu') + F_{1+r+n}(\lambda', \mu')F_{1+r-n}(\lambda, \mu)\}q^{n^2}. \quad (29)$$

The T_r function is similar to this as in (16). I do not pursue these farther. The reader can easily write down identities of the type (25), etc., by using double-pair products of the form

$$\{F_r(\lambda, \mu)F_s(\lambda', \mu') + F_r(\lambda', \mu')F_s(\lambda, \mu)\}$$

in place of the single pairs of (15), (16).

9. Analytical properties of the polynomials $f_n(x)$, $l_n(x)$, $R_n(x)$

I define three polynomials,

$$f_n(x) = (\omega_1^n - \omega_2^n)/(\omega_1 - \omega_2), \qquad l_n(x) = \omega_1^n + \omega_2^n, \\ R_n(x) = (\omega_1^{2^n} + \omega_2^{2^n}), \quad (30)$$

in which $\omega_1 = \frac{1}{2}\{1+\sqrt{(1+4x)}\}$, $\omega_2 = \frac{1}{2}\{1-\sqrt{(1+4x)}\}$. They are analogues of the numbers u_n, v_n, r_n in the Fibonacci and Lucas sequences, and reduce to those numbers when $x = 1$. They have many interesting properties, more especially in connexion with prime numbers. For convenience I now use the variable x in place of the parameter λ used in previous sections of the paper. Dr. Rado, in a paper which he has kindly allowed me to see, has effected the complete resolution of $f_n(x)$ into its factors irreducible in the rational field.

Explicitly, in a terminating series we have

$$l_n(x) = 1 + nx + \frac{n(n-3)}{2!}x^2 + \frac{n(n-4)(n-5)}{3!}x^3 + \ldots \quad (31)$$

in which the final term is $nx^{\frac{1}{2}(n-1)}$ or $2x^{\frac{1}{2}n}$ according as n is odd or even. From the definitions we can obtain at once the simpler results

$$f_m(x)l_m(x) = f_{2m}(x), \qquad 2f_{m+n}(x) = f_m(x)l_n(x) + f_n(x)l_m(x), \\ \frac{dl_n}{dx} = nf_{n-1}(x), \qquad f_m(x)f_n(x) \mid f_{mn}(x) \quad \text{when } (m, n) = 1, \\ R_n(2) = 2^{2^n} + 1 \quad \text{(a Fermat number)}, \\ l_p(2) = 2^p - 1 \quad \text{(a Mersenne number)}, \\ R_{n+1}(x) = R_n^2(x) - 2x^{2^n}.$$

Then by repetition and simple induction a quasi-addition theorem for m, n is

$$R_{m+n}(x) = R_n^{2^m}(x) \times R_m(\xi) \quad (\xi = -x^{2^m}/R_n^2(x)), \tag{32}$$

where m, n are interchangeable. Many other relations of more or less interest can be obtained by simple algebra.

Thus, since $f_m \mid f_n$ when $m \mid n$, and it is well known (**3**) that

$p \mid u_{p-1}$, also $p \mid (u_p - 1)$ if p is a prime of the form $5m \pm 1$,

$p \mid u_{p+1}$, also $p \mid (u_p + 1)$ if p is a prime of the form $5m \pm 2$,

it follows that $f_p \mid f_m$ when $m = u_{p-1}$ if p is a prime of the form $5m \pm 1$. There is a similar theorem for u_{p+1}, when $m = u_{p+1}$ and p is a prime of the form $5m \pm 2$. It is easy to obtain from Fermat's theorem and the ω-formula (30) that

$$f_{p-1}(m^2 \pm m) \equiv 1 \pmod{p} \text{ if } p \nmid (m^2 \pm m). \tag{33}$$

10. Irreducible factors and prime numbers

There is a well-known test of irreducibility for complete integral polynomials in the rational field due to Gauss and Eisenstein (**2**). If in

$$1 + a_1 x + a_2 x^2 + \ldots + a_n x^n, \tag{34}$$

where a_n is a prime number, and

$$(a_1, a_2, \ldots, a_{n-1}) > 0, \qquad a_n \mid a_r \quad (r = 1, 2, 3, \ldots, n-1),$$

then the polynomial is irreducible in the rational field. Rado's resolution of $f_n(x)$ into irreducible complete rational factors (**8**) is

$$\left.\begin{aligned} f_n(x) &= \prod_{m \mid n} H_m(x) \quad (m \leqslant n), \\ \text{in which} \qquad H_m(x) &= 1 + a_1 x + a_2 x^2 + \ldots + a_\kappa x^\kappa, \\ 2\kappa \text{ is Euler's function } \phi(n), &\text{ and} \\ (a_1, a_2, \ldots, a_\kappa) &> 0. \end{aligned}\right\} \tag{35}$$

From these two theorems, (34), (35), many interesting properties of $f_n(x)$, $l_n(x)$, $R_n(x)$ follow at once.

(i) If the rank n be a prime number p, then $f_p(x)$, $l_p(x)$, are irreducibles, while $R_n(x)$ is irreducible for all positive n. Inspection of the explicit forms of $l_p(x)$, $R_n(x)$ shows that both these conform to Gauss's test, while $f_p(x)$ is irreducible under Rado's theorem (35);

(ii) if h be the greatest common divisor of m, n, then $f_h(x)$ is the highest common factor of $f_m(x)$, $f_n(x)$:

here $f_h(x)$ may be a product of two or more irreducible H factors;

(iii) $f_n(x) = 0$ has no repeated roots;

(iv) if in an irreducible $H_m(x)$, $H_m(1)$ is a prime number p, then $H_m(x) \mid f_{p-1}(x) f_{p+1}(x)$. The usual deductions can be obtained according as p is of form $5m \pm 1$ or $5m \pm 2$.

Here we have the direct analogue of $p \mid u_{p-1} u_{p+1}$ in the Fibonacci numerical sequence.

I give here explicitly four typical examples of the $H_m(x)$ irreducibles:

$$H_{26}(x) = 1+13x+65x^2+156x^3+182x^4+91x^5+13x^6,$$

$$H_{26}(1) = 521 \text{ (prime)};$$

$$H_{33}(x) = 1+21x+188x^2+934x^3+2806x^4+5202x^5+ \\ +5809x^6+3629x^7+1090x^8+120x^9+x^{10},$$

$$H_{33}(1) = 19801 \text{ (prime)};$$

$$H_{38}(x) = 1+19x+152x^2+665x^3+1729x^4+2717x^5+ \\ +2508x^6+1254x^7+285x^8+19x^9,$$

$$H_{38}(1) = 9349 \text{ (prime)};$$

$$H_{50}(x) = 1+20x+170x^2+800x^3+2775x^4+4005x^5+ \\ +4300x^6+2675x^7+875x^8+125x^9+5x^{10},$$

$$H_{50}(1) = 101 \times 151 \text{ (composite)}.$$

It will be noticed that $H_{26}(x)$, $H_{38}(x)$, $H_{50}(x)$ conform to Gauss's test for irreducibility. I have obtained explicitly fifty $H_m(x)$ irreducibles, in fact all the irreducibles in the $f_n(x)$ sequence ($n = 1, 2, 3, \ldots, 55$). The final coefficient a_κ in every case (if we include unity) is a prime number. Thirty-seven out of the fifty cases reduced to a single prime when $x = 1$. Twelve reduced to a form $p_1 \times p_2$; one only to form $p_1 p_2 p_3$. As above $H_{50}(1) = 101 \times 151$, composite with 101 a so-called irregular prime; $H_{37}(1) = 73 \times 149 \times 221$, with 149 an irregular prime. When $H_m(1) = p$ a single prime I venture to name $H_m(x)$ a *prime polynomial*. Such polynomials appear to have a special affinity with prime numbers, as the following congruences show. In the 37 cases (explicitly obtained, some of great length) I found without exception that to modulus $H_6(x) = 1+4x+x^2$

$$H_m(x) \equiv \pm x^\kappa, \text{ or } \pm x^{\kappa-1}, \text{ or } \pm x^{\kappa+1}. \tag{36}$$

When $x = 1$, $H_6(1) = 6$, and this corresponds *exactly* to the property of prime integers

$$p \equiv \pm 1 \pmod 6,$$

Finally, I note that Lehmer in the Introduction to his table of primes gives a list of 32 known irregulars as found by Kummer and Vandiver respectively. The list ends with the irregular 607. There is a large gap between the numbers found by Kummer and those found by Vandiver. An examination of *composite* $H_m(1)$, and especially of twenty-nine composite $H_p(1)$ in my lists, shows invariably the presence of an irregular prime. It is possible that every composite $H_m(6)$ and every composite $H_p(1)$ may contain an irregular. Conjecture however is rather unsafe until Lehmer's list of irregulars is greatly extended. Further investigation is called for.

11. Solution of the q^θ-equations

To obtain solutions of the q^θ-equations (4)–(7) I follow the method commonly used in finding explicit solutions of ordinary linear differential equations. I replace the dependent variable y_λ in (4) by a series $\sum A_n x^n$ with undetermined coefficients. In the case of equations (6), (7) the dependent variables are replaced by the formal series $\sum \{C_n x^{2n} + D_n x^{2n+1}\}$. I carry out the operations indicated by q^θ, $q^{2\theta}$,... in the respective equations and equate the resulting series to zero identically, thus obtaining recurrence-formulae which the coefficients A_n, C_n, D_n must necessarily satisfy. The recurrence-formulae are

$$A_n \lambda q^{2n+1} + A_{n+1} q^{n+1} - A_{n+2} = 0, \tag{37}$$

$$\left.\begin{aligned} \lambda^3 C_n q^{6n+6} - C_{n+1}\lambda(1+\lambda)q^{4n+6} - C_{n+2}(1+\lambda)q^{2n+4} + C_{n+3} &= 0 \\ \lambda^3 D_n q^{6n+9} - D_{n+1}\lambda(1+\lambda)q^{4n+8} - D_{n+2}(1+\lambda)q^{2n+5} + D_{n+3} &= 0 \end{aligned}\right\}, \tag{38}$$

$$\left.\begin{aligned} C'_n \lambda^2\mu^2 q^{8n+12} - C_{n+1}\lambda\mu q^{6n+12} - C'_{n+2}(\lambda+\mu+2\lambda\mu)q^{4n+10} - \\ - C'_{n+3} q^{2n+8} + C'_{n+4} = 0 \\ D'_n \lambda^2\mu^2 q^{8n+16} - D'_{n+1}\lambda\mu q^{6n+15} - D'_{n+2}(\lambda+\mu+2\lambda\mu)q^{4n+12} - \\ - D'_{n+3} q^{2n+9} + D'_{n+4} = 0 \end{aligned}\right\}. \tag{39}$$

When we proceed to form a sequence A_0, A_1, A_2,..., A_6, we have no difficulty in conjecturing that

$$A_n = \{c_1 f_{n-1}(\lambda) + c_2 f_n(\lambda)\} q^{\frac{1}{2}n(n-1)}. \tag{40}$$

It is easy to verify that (37) is satisfied by (40).

In the cases (38), (39) the method breaks down because the forms of the sequences C_0, C_1, C_2, C_3,..., C_6, etc., appear so complex and lengthy that it is impracticable to conjecture a form C_n by inspection. Another plan is adopted. Reference to a paper (**5**) in a previous

number of this Journal shows that, when we have the special simpler case ($\lambda = 1$) of the above equations, C_n is of the form

$$\{\rho u_n u_{n-1} + \sigma u_n u_{n-2} + \tau u_{n-1} u_{n-2}\} q^{n(n-1)},$$

ρ, σ, τ being arbitrary constants. Since $f_n(\lambda)$ is a natural generalization of the Fibonacci u_n, it is now easy to conjecture that

$$C_n = \{\rho f_n f_{n-1} + \sigma f_n f_{n-2} + \tau f_{n-1} f_{n-2}\} q^{n(n-1)},$$

with a similar form for D_n, in which q^{n^2} replaces $q^{n(n-1)}$.

Considerations of necessary symmetry in λ, μ, and the necessity for identity between the solutions (10), (11) when $\mu = \lambda$ lead us to the solution of (11) as stated. Verification by substitution is only a matter of tedious elementary algebra.

REFERENCES

1. Dickson, *History of Number*, vol. 1, ch. xvii.
2. Gauss, *Ges. Werke*, vol. 1, 51.
3. Hardy and Wright, *Theory of Numbers* (Oxford, 1938), 223.
4. Jackson, *Quart. J. of Math.* (Oxford), 11 (1940), 1–17.
5. —— ibid. 12 (1941), 211–15.
6. Lucas, *American Jour. of Math.* 3 (1878), 184–239.
7. Macmahon, *Combinatory Analysis* (Cambridge, 1915), vol. 2, 41–7.
8. Rado, see below, pp. 111–15.

THE IRREDUCIBLE FACTORS OF CERTAIN POLYNOMIALS

By R. RADO (*Sheffield*)

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1. F. H. JACKSON† has introduced the polynomials $f_0(x)$, $f_1(x)$,... defined by

$$f_0(x) = 0, \qquad f_1(x) = 1, \tag{1}$$

$$f_n(x) = f_{n-1}(x) + x f_{n-2}(x) \quad (n = 2, 3, \ldots). \tag{2}$$

Here I find the representation of f_n as a product of irreducible polynomials and deduce some properties of these irreducible factors.

THEOREM 1. *There are polynomials $H_m(x)$ $(m = 3, 4, \ldots)$ with rational integral coefficients which are irreducible in the rational field and satisfy*

$$f_n = \prod_{m>3,\, m|n} H_m \quad (n \geqslant 3). \tag{3}$$

If $m \neq m_1$, then H_m and H_{m_1} are coprime.

THEOREM 2. *If $\phi(m)$ is Euler's function, $\mu(m)$ Möbius's function, and if $m \geqslant 3$, then*

$$H_m(x) = c_{m0} + c_{m1} x + \ldots + c_{mk} x^k, \tag{4}$$

where $k = \frac{1}{2}\phi(m)$,

$$c_{m0} = 1, \tag{5}$$

$$c_{m1} = \phi(m) + \mu(m), \tag{6}$$

$$c_{m2} = \tfrac{1}{2}\{\phi(m)+\mu(m)\}^2 - \tfrac{3}{2}\phi(m) + \begin{cases} -\frac{5}{2}\mu(m) & (2 \nmid m), \\ -\frac{3}{2}\mu(m) & (2 \mid m,\ 4 \nmid m), \\ -\mu(\frac{1}{2}m) & (4 \mid m), \end{cases} \tag{7}$$

$$c_{mk} = \begin{cases} p & (m = 2p^\alpha,\ \alpha > 0,\ p \text{ a prime} \geqslant 2), \\ 1 & (\textit{otherwise}), \end{cases} \tag{8}$$

$$c_{m\kappa} > 0 \quad (0 \leqslant \kappa \leqslant k). \tag{9}$$

When $m = 3$, 4, or 6, $k = 1$ and $c_{m2} = 0$ by (7); when $m = 5$, 8, or 10, $k = 2$ and $c_{m,2} = c_{m,k}$ by (7) and (8). For all other values of m greater than 3, $k > 2$.

2. We take x real and positive and write

$$u = \tfrac{1}{2}\{1+\sqrt{(1+4x)}\}, \qquad v = \tfrac{1}{2}\{1-\sqrt{(1+4x)}\} \tag{10}$$

so that u, v are the roots of the equation $T^2 = T + x$. Then

$$u+v = 1, \qquad uv = -x \tag{11}$$

† See above, pp. 99–110.

and $$f_n(x) = \frac{u^n - v^n}{u - v}$$

since the latter function satisfies (1) and (2).

Let $F_m(y)$ denote the mth cyclometric polynomial: that is

$$F_m(y) = \prod_{\substack{1 \leqslant \nu \leqslant m \\ (\nu, m) = 1}} (y - e^{2\pi i \nu/m}),$$

so that $$y^n - 1 = \prod_{m|n} F_m(y)$$

by a well-known property of $F_m(y)$. Hence

$$f_n(x) = \frac{v^n}{u-v}\left\{\left(\frac{u}{v}\right)^n - 1\right\} = \frac{1}{u-v}\prod_{m|n}\left\{v^{\phi(m)} F_m\left(\frac{u}{v}\right)\right\}$$
$$= (u-v)^{-1}\prod_{m|n} H_m(x),$$

where $$H_m(x) = v^{\phi(m)} F_m\left(\frac{u}{v}\right) \quad (m > 0). \tag{12}$$

Since $H_1 = u - v$, $H_2 = u + v = 1$, we have expressed $f_n(x)$ in the form (3). It remains to show that $H_m(x)$ has the properties stated.

I first show that $H_m(x)$ is a polynomial with rational integral coefficients. Henceforth we take $m \geqslant 3$, so that $l = \phi(m)$ is even, and write $\epsilon = e^{2\pi i/m}$. Throughout we use ν to denote a number such that $1 \leqslant \nu < m$, $(\nu, m) = 1$, and $\sum_\nu$ to denote summation over all such values of ν. Then

$$H_m(x) = v^l \prod_\nu \left(\frac{u}{v} - \epsilon^\nu\right) = \prod (u - v\epsilon^\nu) = \prod (v\epsilon^\nu - u)$$
$$= \epsilon^s \prod (v - u\epsilon^{-\nu}) = \epsilon^s u^l \prod \left(\frac{v}{u} - \epsilon^\nu\right),$$

where $$s = \sum_\nu \nu = \sum_{\nu < \frac{1}{2}m} \{\nu + (m - \nu)\} = \tfrac{1}{2} l m$$

is divisible by m. Hence $\epsilon^s = 1$, and

$$H_m(x) = \prod (u - v\epsilon^\nu) = \prod (v - u\epsilon^\nu).$$

Thus H_m is a symmetrical polynomial in u, v with rational integral coefficients, and therefore, in view of (11), a polynomial in x with rational integral coefficients.

3. Since

$$H_m = \prod_{\nu < \frac{1}{2}m} \{(u - v\epsilon^\nu)(u - v\epsilon^{m-\nu})\} = \prod_{\nu < \frac{1}{2}m} (1 + \alpha_\nu x), \tag{13}$$

where $\alpha_\nu = 4\cos^2(\pi\nu/m) > 0$ for $1 \leqslant \nu < \frac{1}{2}m$, we have (4), (5), and (9).

4. If H_m were reducible in the rational field, we should have, identically in x,

$$H_m(x) = A(x)B(x),$$

where A and B are polynomials with rational coefficients of degree α, β respectively and

$$\alpha > 0, \qquad \beta > 0, \qquad \alpha+\beta = k.$$

If we write

$$y = \frac{u}{v} = \frac{1+\sqrt{(1+4x)}}{1-\sqrt{(1+4x)}},$$

we have

$$x = -y(y+1)^{-2}, \qquad v = (y+1)^{-1},$$

$$(y+1)^{-l}F_m(y) = v^l F_m\left(\frac{u}{v}\right) = H_m(x) = A(x)B(x),$$

$$F_m(y) = A_1(y)B_1(y),$$

where

$$A_1(y) = (y+1)^{2\alpha}A\{-y(y+1)^{-2}\},$$
$$B_1(y) = (y+1)^{2\beta}B\{-y(y+1)^{-2}\}$$

are polynomials in y with rational coefficients. Since $F_m(y)$ is an irreducible polynomial, either $A_1(y)$ or $B_1(y)$ is a constant; let us suppose that

$$A_1(y) = C, \qquad A(x) = Cv^{2\alpha},$$

where C is a constant.

If we now put $x = 0$, we have $v = 0$ by (10) and so, since $\alpha > 0$, $A(0) = 0$ and $H_m(0) = 0$. But, by (13), $H_m(0) = 1$, a contradiction, and so $H_m(x)$ is irreducible.

5. It follows that, if $H_m(x)$ and $H_{m_1}(x)$ are not coprime, one is a constant multiple of the other. But $H_m(0) = H_{m_1}(0) = 1$ and so $H_m(x) = H_{m_1}(x)$ and, in particular, they are of the same degree. Hence

$$F_m\left(\frac{u}{v}\right) = F_{m_1}\left(\frac{u}{v}\right)$$

and so $m = m_1$. This completes the proof of Theorem 1.

6. It remains to prove (6), (7), and (8). It is well known that $\sum \epsilon^\nu = \mu(m)$. Hence, by (13),

$$c_{m1} = \sum_{\nu<\frac{1}{2}m} \alpha_\nu = \sum_{\nu<\frac{1}{2}m} (2+\epsilon^\nu+\epsilon^{-\nu}) = 2 \cdot \tfrac{1}{2}\phi(m) + \sum_\nu \epsilon^\nu = \phi(m)+\mu(m).$$

This is (6).

The numbers $\epsilon^{2\nu}$ are primitive roots of unity of order $m' = m/(2, m)$.

Hence every root of the polynomial

$$g(x) = \prod_{\nu} (x - \epsilon^{2\nu})$$

is a root of the irreducible polynomial $f(x) = F_{m'}(x)$, and therefore, since both f and g have their highest coefficient equal to unity,

$$\prod_{\nu} (x - \epsilon^{2\nu}) = \{F_{m'}(x)\}^h, \tag{14}$$

where $$h = \phi(m)/\phi(m').$$

Hence the $\phi(m)$ numbers $\epsilon^{2\nu}$ are the $\phi(m')$ primitive m'th roots of unity, each repeated h times. In particular,

$$\sum_{\nu} \epsilon^{2\nu} = h\mu(m').$$

Therefore, by (13),

$$2c_{m2} = 2 \sum_{\nu<\nu'<\frac{1}{2}m} \alpha_\nu \alpha_{\nu'} = \Big(\sum_{\nu<\frac{1}{2}m} \alpha_\nu\Big)^2 - \Big(\sum_{\nu<\frac{1}{2}m} \alpha_\nu^2\Big)$$

$$= c_{m1}^2 - \sum_{\nu<\frac{1}{2}m} (4 + \epsilon^{2\nu} + \epsilon^{-2\nu} + 4\epsilon^{\nu} + 4\epsilon^{-\nu} + 2)$$

$$= c_{m1}^2 - 3\phi(m) - \sum_{\nu} (4\epsilon^{\nu} + \epsilon^{2\nu}) = c_{m1}^2 - 3\phi(m) - 4\mu(m) - h\mu(m').$$

If $2 \nmid m$, then $m' = m$ and $h = 1$. If $2 \mid m$, $4 \nmid m$, then $m' = \frac{1}{2}m$, $h = 1$, $\mu(m') = -\mu(m)$.

If $4 \mid m$, then $m' = \frac{1}{2}m$, $h = 2$, $\mu(m) = 0$.

This establishes (7).

Finally we have to calculate c_{mk}. By (13) and (9),

$$c_{mk} = |c_{mk}| = \prod_{\nu<\frac{1}{2}m} |\alpha_\nu| = \prod_{\nu<\frac{1}{2}m} |\epsilon^{\frac{1}{2}\nu} + \epsilon^{-\frac{1}{2}\nu}|^2$$

$$= \prod_{\nu<\frac{1}{2}m} |\epsilon^{\nu} + 1|^2 = \prod_{\nu} |\epsilon^{\nu} + 1| = |F_m(-1)|.$$

Now, by (14),

$$F_m(-1) = \prod (-1 - \epsilon^{\nu}) = \{\prod (1 - \epsilon^{2\nu})\}\{\prod (1 - \epsilon^{\nu})\}^{-1}$$

$$= \{F_{m'}(1)\}^h \{F_m(1)\}^{-1}. \tag{15}$$

We have $m' \geqq \frac{1}{2}m > 1$. For any $n > 1$

$$F_n(1) = \lim_{x \to 1} F_n(x) = \lim \prod_{d \mid n} (x^{n/d} - 1)^{\mu(d)}$$

$$= \lim \prod_{d \mid n} \left(\frac{x^{n/d} - 1}{x - 1}\right)^{\mu(d)} = \prod_{d \mid n} \left(\frac{n}{d}\right)^{\mu(d)} = \prod_{d \mid n} d^{-\mu(d)},$$

since $$\sum_{d \mid n} \mu(d) = 0.$$

Suppose that $n = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_r^{\alpha_r}$, where

$$r \geqq 1, \qquad p_1 < p_2 < \ldots < p_r \qquad \alpha_1, \alpha_2, \ldots, \alpha_r > 0.$$

The p_ρ are primes. Then

$$\prod_{d|n} d^{-\mu(d)} = \prod_{0\leqslant\lambda_1,\lambda_2,\ldots,\lambda_r\leqslant 1} (p_1^{\lambda_1}\ldots p_r^{\lambda_r})(-1)^{\lambda_1+\ldots+\lambda_r-1} = \prod_\rho p_\rho^{\beta_\rho}$$

where
$$\beta_1 = \beta_2 = \ldots = \beta_r = \sum_{\substack{\lambda_1=1\\ 0\leqslant\lambda_2,\ldots,\lambda_r\leqslant 1}} (-1)^{\lambda_1+\ldots+\lambda_r-1}$$

$$= 1-\binom{r-1}{1}+\binom{r-1}{2}-+\ldots,$$

as is seen by summing the last sum over the various values of $\lambda_2+\ldots+\lambda_r$.

It follows that

$$F_n(1) = p_1 \quad \text{if } r = 1, \qquad F_n(1) = 1 \quad \text{if } r > 1. \tag{16}$$

We now verify (8). It is obvious from (15), (16) that $c_{mk} \neq 1$ is only possible when $m = 2p^\alpha$, where p is prime and $\alpha > 0$. If now $p = 2$, then $m' = \frac{1}{2}m$, $h = 2$,

$$F_m(1) = F_{m'}(1) = 2, \qquad c_{mk} = 2^2\,2^{-1} = 2.$$

On the other hand, if $p > 2$, then $m' = \frac{1}{2}m$, $h = 1$,

$$F_m(1) = 1, \qquad F_{m'}(1) = p, \qquad c_{mk} = p^1\,1 = p.$$

This completes the proof of (8) and hence the proofs of Theorems 1 and 2.

7. The identity (14), without the restriction $m \geqslant 3$, generalizes at once to

$$\prod_\nu (x-\epsilon^{j\nu}) = \{F_{m^*}(x)\}^{h^*}, \tag{17}$$

where j is any positive integer, $m^* = m/(m,j)$, $h^* = \phi(m)/\phi(m^*)$ and thus leads to an evaluation of Ramanujan's sum

$$c_m(j) = \sum_{\substack{1\leqslant\nu\leqslant m\\ (\nu,m)=1}} e^{2\pi i\nu j/m}$$

in the form
$$c_m(j) = h^*\mu(m^*),$$

which can be seen to be equivalent to the more usual form.† It is easy to prove (17) without making use of the irreducibility of the cyclometric polynomials.

From the proofs in § 6 it is clear that, for any fixed positive value of κ, the coefficient $c_{m\kappa}$ can be expressed in terms of the functions ϕ and μ.

I should like to thank the referee for his suggestions.

† See, for instance, Hardy and Wright, *An Introduction to the Theory of Numbers*, 237, Theorem 271.

ON CERTAIN TAC-INVARIANTS OF TWO CURVES IN A PROJECTIVE SPACE

By B. SU (*Meitan*)

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As an extension of a theorem of B. Segre, I have recently shown that two curves C and Γ in a projective space S_n ($n \geqslant 3$), having at a common point P the same osculating spaces of dimensions 1, 2,..., r ($1 \leqslant r \leqslant n-1$), admit at P certain $r-1$ projective invariants.* In a very recent paper B. Segre has proved this result afresh in an extremely simple way and also obtained $n-r-1$ new projective invariants.† Here I give an alternative proof of my previous result; this proof, even if not so simple as Segre's, leads to an interesting property of some plane curves determined by the developable hypersurfaces of C and Γ.

We introduce non-homogeneous coordinates $(x_1, x_2,..., x_n)$, in such a way that P is the origin, and the common p-dimensional osculating space of C and Γ at P, say S_p, has the equations

$$x_{p+1} = x_{p+2} = \ldots = x_n = 0 \quad (1 \leqslant p \leqslant r).$$

Then C and Γ have the respective parametric equations

$$x_1 = t, \qquad i!\,x_i = \sum_{l=0}^{\infty} a_{i,i+l} t^{i+l}, \qquad j!\,x_j = \sum_{l=0}^{\infty} a_{j,r+l+1} t^{r+l+1}, \tag{1}$$

and

$$x_1 = t, \qquad i!\,x_i = \sum_{l=0}^{\infty} \alpha_{i,i+l} t^{i+l}, \qquad j!\,x_j = \sum_{l=0}^{\infty} \alpha_{j,r+l+1} t^{r+l+1}, \tag{2}$$

where $i = 2,..., r$ and $j = r+1,..., n$.

The $(n-2)$-dimensional space osculating C at a general point meets S_2 at a point whose locus is a curve, C_2 say, containing P. It is clear that, if $n = 3$, $r = 2$, the section of S_2 and the developable surface of C consists of C_2 and the line S_1; if $n > 3$, $r \geqslant 2$, the developable hypersurface of C contains S_2, and C_2 is for it a locus of points of higher multiplicity than the general point of S_2. In a similar way, the developable hypersurface of Γ determines on S_2 a curve Γ_2. If we introduce on S_2 homogeneous coordinates ξ_0, ξ_1, ξ_2,

* B. Su, 'Note on a theorem of B. Segre': *Science Record, Ac. Sinica*, 1 (1942), 16–19.

† B. Segre, 'On tac-invariants of two curves in a projective space': see above, pp. 35–8.

by putting $x_1 = \xi_1/\xi_0$, $x_2 = \xi_2/\xi_0$, we obtain for C_2 and Γ_2 the parametric equations

$$\xi_0 = [x_3', x_4', \ldots, x_n'], \quad \xi_1 = [x_1, x_3, x_4, \ldots, x_n], \quad \xi_2 = [x_2, x_3, \ldots, x_n];$$

here the x are the functions (1) and (2) respectively, the x' are their derivatives, and the brackets [ ] denote the Wronskian of the arguments enclosed.* A simple calculation then gives the equations of C_2 in the parametric form

$$(n-2)!\,(n-1)!\,\xi_0 = a_{33}\ldots a_{rr}\,At^{2n-4}+\ldots,$$
$$\tfrac{1}{2}(n-2)!\,n!\,\xi_1 = a_{33}\ldots a_{rr}\,At^{2n-3}+\ldots,$$
$$(n-1)!\,n!\,\xi_2 = a_{33}\ldots a_{rr}\,At^{2n-2}+\ldots,$$

where
$$A = \begin{vmatrix} a_{r+1,r+1} & \cdot & \cdot & a_{r+1,n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n,r+1} & \cdot & \cdot & a_{nn} \end{vmatrix},$$

whence we deduce $$x_2 = \frac{n}{4(n-1)}a_{22}x_1^2+\ldots.$$

Likewise the equation of Γ_2 is

$$x_2 = \frac{n}{4(n-1)}\alpha_{22}x_1^2+\ldots.$$

Hence the invariant of contact of C_2 and Γ_2 at P is

$$J_2 = a_{22}/\alpha_{22},$$

and gives a first tac-invariant of C and Γ at P.

We now suppose $r \geqslant 3$ and, taking any plane π_m which passes through S_1 and lies in S_m but not in S_{m-1} $(3 \leqslant m \leqslant r)$, we consider the curves C_m, Γ_m determined on it as before by the developables of C, Γ. The equations of π_m can be assumed to be

$$x_2+\lambda_2 x_m = 0, \quad \ldots, \quad x_{m-1}+\lambda_{m-1}x_m = 0, \quad x_{m+1} = 0, \quad \ldots, \quad x_n = 0,$$

and so C_m, Γ_m are represented on π_m by the homogeneous parametric equations

$$\xi_0 = [x_2'+\lambda_2 x_m', \ldots, x_{m-1}'+\lambda_{m-1}x_m', x_{m+1}', \ldots, x_n'],$$
$$\xi_1 = [x_1, x_2+\lambda_2 x_m, \ldots, x_{m-1}+\lambda_{m-1}x_m, x_{m+1}, \ldots, x_n],$$
$$\xi_m = (-1)^m[x_2, x_3, \ldots, x_n],$$

where the symbols have the same meaning as before.† It is remark-

* Cf. B. Su, 'Contributions to the projective theory of curves in space of n dimensions' (1st Memoir): *Journ. Chinese Math. Soc.* 2 (1940), 153–73.

† Cf. B. Su, 'Contributions to the projective theory of curves in space of n dimensions' (2nd Memoir): ibid. 277–89.

able that, in the expansion of the Wronskians on the right-hand sides, the parameters λ do not appear in the terms of lowest degree in t. In fact, by a simple calculation, we deduce for C_m

$$\xi_0 = \frac{m-1}{(r-m)!\,(n-1)!}\, a_{22}\ldots a_{m-1,\,m-1}\, a_{m+1,\,m+1}\ldots a_{rr}\, At^{2n-m-2}+\ldots$$

$$\xi_1 = \frac{m}{n!\,(n-m)!}\, a_{22}\ldots a_{m-1,\,m-1}\, a_{m+1,\,m+1}\ldots a_{rr}\, At^{2n-m-1}+\ldots$$

$$\xi_m = \frac{(-1)^m}{n!\,(n-1)!}\, a_{22}\ldots a_{rr}\, At^{2n-2}+\ldots,$$

so that the equation of C_m on π_m is

$$x_m = (-1)^m \frac{(n-m)!}{n!} \left\{\frac{(m-1)(n-m)!}{(r-m)!}\right\}^{m-1} a_{mm}\, x_1^m+\ldots.$$

Likewise the equation of Γ_m on π_m is

$$x_m = (-1)^m \frac{(n-m)!}{n!} \left\{\frac{(m-1)(n-m)!}{(r-m)!}\right\}^{m-1} \alpha_{mm}\, x_1^m+\ldots.$$

Hence the invariant of contact of C_m and Γ_m at P is

$$J_m = a_{mm}/\alpha_{mm} \quad (m = 3,\ldots, r)$$

and is independent of the parameters λ, so that J_m is a tac-invariant of C and Γ at P.

It should be noticed that, on applying the process above for $r+1 \leqslant m \leqslant n$, we obtain no tac-invariant of C and Γ, since now the tac-invariant of C_m and Γ_m depends on the parameters λ. However, on imposing on the latter invariant some special conditions, we may define sets of planes π_m which are projective covariants of C and Γ at P.

THEORY OF A RULED TWO-SPACE IN A GENERALIZED METRIC SPACE

By J. G. FREEMAN (*Wolverhampton*)

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1. Preliminary

S_n is an n-dimensional metric space,* with coordinates x^i ($h, i, j, k...$ run from 1 to n throughout), to each point of which is associated a contravariant vector-density of weight p, called the *element of support*. The unit vector in the direction of the element of support has components l^i.

S_2 is a 2-space in S_n, with coordinates x^α ($\alpha, \beta, \gamma...$ have values 1, 2 throughout), the same letter x being used for coordinates both of S_n and S_2 since the index $i...$ or $\alpha...$ indicates to which space we are referring.

If there exists in S_2 a one-parameter family of curves such that each member of the family is an extremal in S_n with the property that the first variation of the length integral vanishes for arbitrary displacement of curve and element of support, then S_2 is said to be a *ruled 2-space in* S_n, and the members of the family of extremals are called the *generators of the* S_2. The parameter of the family will be denoted by t.

I here consider only the case in which

(i) *the element of support is tangential to each generator.*†

Necessary and sufficient conditions‡ for the generators to be such extremals and for (i) to be satisfied are

(ii) *each generator must be autoparallel in* S_n;

(iii) *the vector* A^i§ *of* S_n *must vanish along each generator.*

If D is any orthogonal trajectory of the generators, the arc-length measured along a generator from its point of intersection with D will be denoted by v.

* As treated by Schouten and Haantjes in (**5**).

† That there are extremals of other types is shown in (**4**), § 4.

‡ For proof see (**4**), § 4.

§ For definition of A^i see (**4**), § 1. It possesses the property that

$$l^j A_{j\,k}^{\,i} = p l^i A_k,$$

proved in (**1**).

I write

$$\left.\begin{aligned} \partial x^i/\partial v = \mu^i, \qquad & \partial x^\alpha/\partial v = \mu^\alpha \\ \partial x^i/\partial t = \lambda^i, \qquad & \partial x^\alpha/\partial t = \lambda^\alpha \end{aligned}\right\} \tag{1.1}$$

for the S_n and S_2 components of vectors tangential to generators $t =$ constant and curves $v =$ constant.

The vector μ^i is a unit vector (v being the arc-length measured along a generator); λ^i in general is not a unit vector and its length will be denoted by λ. The unit vector in the direction of λ^i having ξ^i, ξ^α for S_n, S_2 components respectively is given by

$$\xi^i = \lambda^i/\lambda, \qquad \xi^\alpha = \lambda^\alpha/\lambda \tag{1.2}$$

provided that $\lambda \neq 0$; when $\lambda = 0$, the vector ξ^i is not defined. Since μ^i is a unit vector tangential to a generator, by (i) above

$$\mu^i = \pm l^i. \tag{1.3}$$

I write also

$$\partial x^i/\partial x^\alpha = B^i_\alpha, \tag{1.4}$$

whilst, if $g_{ij}\,dx^i dx^j$, $g_{\alpha\beta}\,dx^\alpha dx^\beta$ are the first fundamental forms of S_n and S_2,

$$g_{\alpha\beta} = g_{ij}\,B^i_\alpha\,B^j_\beta. \tag{1.5}$$

The tensor $g^{\alpha\beta}$ is given by

$$g^{\alpha\beta}g_{\alpha\gamma} = \delta^\beta_\gamma, \tag{1.6}$$

and I put

$$g^{\alpha\beta}g_{ij}\,B^i_\alpha = B^\beta_j. \tag{1.7}$$

If a vector tangential to S_2 has S_2 components X^α and S_n components $X^i = B^i_\alpha X^\alpha$, then DX^α (the absolute derivative of X^α induced in S_2) is given by the S_2 components of the projection on S_2 of DX^i (the absolute derivative of X^i in S_n). Thus

$$DX^\alpha = B^\alpha_i\,DX^i; \tag{1.8}$$

DX^i, DX^α are of the form*

$$DX^i = dX^i + X^j\omega_j{}^i, \qquad DX^\alpha = dX^\alpha + X^\beta\omega_\beta{}^\alpha. \tag{1.9}$$

Defining,† for an expression T^i_α involving indices of both species,

$$DT^i_\alpha = dT^i_\alpha + T^j_\alpha\,\omega_j{}^i - T^i_\beta\,\omega_\alpha{}^\beta, \tag{1.10}$$

we see that the rule for operating on a product by D is the same as that for ordinary differentiation of a product; e.g.

$$D(B^i_\alpha X^\alpha) = DB^i_\alpha X^\alpha + B^i_\alpha DX^\alpha. \tag{1.11}$$

Further,

$$g_{ij}\,B^i_\alpha\,DB^j_\beta = 0. \tag{1.12}$$

* Values of $\omega_j{}^i$, $\omega_\beta{}^\alpha$ are given in (**3**), §§ 5, 15; they are such that Dg_{ij}, $Dg_{\alpha\beta} = 0$; the form of $\omega_j{}^i$ is required in this work only in the evaluation of $D\lambda^i/\partial v - D\mu^i/\partial t$ which follows.

† Following (**3**), § 15, in which are given proofs of (1.11), (1.12).

Now
$$\frac{\partial}{\partial v}(\lambda^i\mu_i) = \frac{D}{\partial v}(\lambda^i\mu_i) = \frac{D\lambda^i}{\partial v}\mu_i,$$

since, along a generator, $D\mu^i/\partial v = 0$ from (ii). But*

$$\frac{D\lambda^i}{\partial v} - \frac{D\mu^i}{\partial t} = \Omega^i = \left(\lambda^j\frac{Dl^k}{\partial v} - \mu^j\frac{Dl^k}{\partial t}\right)A_j{}^i{}_k,$$

whilst $\mu^j A_j{}^i{}_k = \pm l^j A_j{}^i{}_k = \pm p l^i A_k = 0$ from (iii); also $Dl^k/\partial v = 0$ from (ii) and (1.3). Thus $\Omega^i = 0$ and

$$\frac{D\lambda^i}{\partial v} = \frac{D\mu^i}{\partial t}, \tag{1.13}$$

whilst, on account of the vanishing of A_k, the results which follow are independent of the weight p. Hence

$$\frac{\partial}{\partial v}(\lambda^i\mu_i) = \frac{D\mu^i}{\partial t}\mu_i = 0,$$

since μ^i is a unit vector. Thus, along a generator, $\lambda^i\mu_i$ is constant; but, at the point of intersection of the generator with D, it vanishes, and hence it vanishes all along the generator. Hence

$$\lambda^i\mu_i = 0 \tag{1.14}$$

at all points of S_2.

I now take v, t for coordinates in S_2, writing

$$x^1 = t, \qquad x^2 = v. \tag{1.15}$$

It follows that, for μ^α and λ^α,

$$\left.\begin{aligned} \mu^1 &= 0, & \lambda^1 &= 1 \\ \mu^2 &= 1, & \lambda^2 &= 0 \end{aligned}\right\}, \tag{1.16}$$

and
$$B_1^i = \lambda^i, \qquad B_2^i = \mu^i. \tag{1.17}$$

Since $g_{\alpha\beta}\lambda^\alpha\mu^\beta = g_{ij}\lambda^i\mu^j = \lambda^i\mu_i = 0$ from (1.14), using (1.16) gives $g_{12} = 0$. Further, since $v = x^2$ is the arc-length along a generator $t = x^1 =$ constant, $(dv)^2 = g_{22}(dv)^2$ and $g_{22} = 1$. Also

$$g = |g_{\alpha\beta}| = \begin{vmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{vmatrix} = g_{11}.$$

Hence
$$g_{11} = g, \qquad g_{12} = 0, \qquad g_{22} = 1. \tag{1.18}$$

Points for which g vanishes are such that $g^{\alpha\beta}$, B_i^α, and DX^α as given by (1.6), (1.7), (1.8) are not determined. Since

$$\lambda^2 = g_{\alpha\beta}\lambda^\alpha\lambda^\beta = g_{11}$$

* See (**4**), § 3.

from (1.16), it follows that

$$\lambda = \sqrt{g_{11}} = \sqrt{g}. \tag{1.19}$$

Points with vanishing g occur if the generators have an envelope, for then, for displacement with variation of t of the point of contact of a generator with the envelope,

$$\frac{dx^i}{dt} = \frac{\partial x^i}{\partial x^\alpha}\frac{dx^\alpha}{dt} = \lambda^i + \mu^i\frac{dv}{dt} = c\mu^i \text{ say},$$

since this displacement is in the direction of the generator at the point of contact. Multiplying by μ_i gives $dv/dt = c$ since $\lambda^i\mu_i = 0$, whence $\lambda^i = 0$ and $\lambda = 0$. Thus

If the generators have an envelope, g vanishes at the points of contact of the generators with the envelope. (1.20)

Since $D\mu^i/\partial v = 0$ from (ii), and $\mu^i = B^i_\alpha\mu^\alpha$, we have, from (1.11),

$$\frac{DB^i_\alpha}{\partial v}\mu^\alpha + B^i_\alpha\frac{D\mu^\alpha}{\partial v} = 0 \tag{1.21}$$

except at points for which g vanishes, when $D\mu^\alpha/\partial v$ is not defined. Multiplying by $g_{ij}B^j_\beta$ gives, using (1.12),

$$g_{\alpha\beta}\frac{D\mu^\alpha}{\partial v} = 0,$$

whence

$$\frac{D\mu^\alpha}{\partial v} = 0 \textit{ provided that } g \neq 0. \tag{1.22}$$

2. Curvature of a ruled S_2

For any vector tangential to S_2 with S_n and S_2 components X^i, X^α respectively,

$$\frac{DX^i}{\partial t} = \frac{D}{\partial t}(B^i_\alpha X^\alpha) = \frac{DB^i_\alpha}{\partial t}X^\alpha + B^i_\alpha\frac{DX^\alpha}{\partial t}$$

from (1.11). Hence, brackets [ ] indicating that from the expression contained therein we subtract the corresponding expression obtained by interchange of t, v,

$$\left[\frac{D}{\partial v}\frac{D}{\partial t}\right]X^i = \left[\frac{D}{\partial v}\frac{DB^i_\alpha}{\partial t}\right]X^\alpha + B^i_\alpha\left[\frac{D}{\partial v}\frac{D}{\partial t}\right]X^\alpha.$$

If we write*

$$\left[\frac{D}{\partial v}\frac{D}{\partial t}\right]X^i = \Omega_j{}^i X^j, \qquad \left[\frac{D}{\partial v}\frac{D}{\partial t}\right]X^\alpha = \Omega_\beta{}^\alpha X^\beta,$$

* The curvature tensor $\Omega_j{}^i$ being defined as in (4), § 2.

it follows that

$$\Omega_j{}^i B^j_\alpha X^\alpha = \left[\frac{D}{\partial v}\frac{DB^i_\alpha}{\partial t}\right] X^\alpha + B^i_\beta \Omega_\alpha{}^\beta X^\alpha.$$

Since the X^α are arbitrary,

$$\Omega_j{}^i B^j_\alpha = \left[\frac{D}{\partial v}\frac{DB^i_\alpha}{\partial t}\right] + B^i_\beta \Omega_\alpha{}^\beta.$$

Multiplying by $g_{ih} B^h_\gamma$ and using

$$g_{ih}\frac{DB^h_\gamma}{\partial v}\frac{DB^i_\alpha}{\partial t} + g_{ih} B^h_\gamma \frac{D}{\partial v}\frac{DB^i_\alpha}{\partial t} = 0$$

obtained by differentiation of (1.12), we get

$$\Omega_{\alpha\gamma} = \Omega_{jh} B^j_\alpha B^h_\gamma + g_{jh}\left[\frac{DB^h_\gamma}{\partial v}\frac{DB^j_\alpha}{\partial t}\right] \tag{2.1}$$

for the Gauss equation to the S_2.* Thus

$$\Omega_{12} = \Omega_{jh}\lambda^j\mu^h + g_{jh}\left[\frac{DB^j_1}{\partial t}\frac{DB^h_2}{\partial v}\right], \tag{2.2}$$

whilst Ω_{11} and Ω_{22} vanish identically since $\Omega_{ij} + \Omega_{ji} = 0$.

Now (1.21) becomes, on using (1.22) and (1.16),

$$\frac{DB^i_2}{\partial v} = 0. \tag{2.3}$$

Hence (2.2) becomes

$$\Omega_{12} = \Omega_{jh}\lambda^j\mu^h - g_{jh}\frac{DB^j_1}{\partial v}\frac{DB^h_2}{\partial t}. \tag{2.4}$$

Since $D\lambda^i/\partial v = D\mu^i/\partial t$ from (1.13), and $\lambda^i = B^i_\alpha\lambda^\alpha$, $\mu^i = B^i_\alpha\mu^\alpha$,

$$\frac{DB^i_\alpha}{\partial v}\lambda^\alpha + B^i_\alpha\frac{D\lambda^\alpha}{\partial v} = \frac{DB^i_\alpha}{\partial t}\mu^\alpha + B^i_\alpha\frac{D\mu^\alpha}{\partial t},$$

i.e.
$$\frac{DB^i_1}{\partial v} + B^i_\alpha\frac{D\lambda^\alpha}{\partial v} = \frac{DB^i_2}{\partial t} + B^i_\alpha\frac{D\mu^\alpha}{\partial t} \quad \text{from (1.16).}$$

Multiplying by $g_{ij}(DB^j_1/\partial v)$ and using (1.12) we have

$$g_{ij}\frac{DB^i_1}{\partial v}\frac{DB^j_1}{\partial v} = g_{ij}\frac{DB^i_2}{\partial t}\frac{DB^j_1}{\partial v},$$

and (2.4) may be written

$$\Omega_{12} = \Omega_{ij}\lambda^i\mu^j - g_{ij}\frac{DB^i_1}{\partial v}\frac{DB^j_1}{\partial v}. \tag{2.5}$$

* As proved in (3), § 16.

Since $\xi^\alpha\xi_\alpha = 1$, then $\xi^\alpha D\xi_\alpha/\partial v = 0$, i.e. $\lambda^\alpha D\xi_\alpha/\partial v = 0$ and $D\xi_1/\partial v = 0$ from (1.16). Since also $\mu^\alpha\lambda_\alpha = 0$ from (1.14), then $\mu^\alpha\xi_\alpha = 0$; therefore

$$\frac{D\mu^\alpha}{\partial v}\xi_\alpha + \mu^\alpha\frac{D\xi_\alpha}{\partial v} = 0;$$

but $D\mu^\alpha/\partial v = 0$ from (1.22), and hence $D\xi_2/\partial v = 0$ by using (1.16). Thus $D\xi_\alpha/\partial v = 0$, whence

$$\frac{D\xi^\alpha}{\partial v} = 0. \tag{2.6}$$

Now $\xi^i = B^i_\alpha\xi^\alpha$, whence

$$\frac{D\xi^i}{\partial v} = \frac{DB^i_\alpha}{\partial v}\xi^\alpha + B^i_\alpha\frac{D\xi^\alpha}{\partial v}.$$

Using (2.6), (1.2) and (1.16) gives

$$\frac{D\xi^i}{\partial v} = \frac{DB^i_\alpha}{\partial v}\frac{\lambda^\alpha}{\lambda} = \frac{1}{\lambda}\frac{DB^i_1}{\partial v},$$

whence, from (1.19),

$$\frac{DB^i_1}{\partial v} = \sqrt{g}\frac{D\xi^i}{\partial v}. \tag{2.7}$$

Equation (2.5) now becomes

$$\Omega_{12} = \Omega_{ij}\lambda^i\mu^j - g(\xi_v)^2, \tag{2.8}$$

where ξ_v is length of vector $D\xi^i/\partial v$.

If, in particular, S_n is a Riemannian space, then

$$\Omega_{ij} = R_{ijhk}\lambda^h\mu^k, \qquad \Omega_{12} = R_{12\alpha\beta}\lambda^\alpha\mu^\beta = R_{1212} = Kg,$$

where K is the total curvature of S_2, whilst, in the particular case in which S_n is Euclidean, $R_{ijhk} = 0$. Hence

$$\begin{array}{|ll|}\hline \textit{In generalized } S_n, & \Omega_{12}/g = \Omega_{ij}\lambda^i\mu^j/g - (\xi_v)^2; \\ \textit{in Riemannian } S_n, & K = R_{ijhk}\xi^i\mu^j\xi^h\mu^k - (\xi_v)^2; \\ \textit{in Euclidean } S_n, & K = -(\xi_v)^2. \\ \hline\end{array} \tag{2.9}$$

Throughout this paragraph points for which g vanishes must be excluded, for at such points the induced absolute derivative and the tensor $\Omega_{\alpha\beta}$ are not defined.

3. Rotation of tangent two-plane

Let η^i be a unit vector normal to a generator at a given point P. This vector is transported by parallelism in S_n along the generator, to which it remains normal since the generator is autoparallel in S_n. If ϕ is the angle between η^i and ξ^i, $\partial\phi/\partial v$ measures the rate of rotation of the tangent 2-plane as P moves along the generator.

Now
$$\cos\phi = \eta^i\xi_i.$$

Hence
$$-\frac{\partial\phi}{\partial v}\sin\phi = \frac{D}{\partial v}(\eta^i\xi_i) = \eta_i\frac{D\xi^i}{\partial v}$$

since $D\eta^i/\partial v = 0$, and

$$-\left(\frac{\partial\phi}{\partial v}\right)^2\cos\phi - \frac{\partial^2\phi}{\partial v^2}\sin\phi = \eta_i\frac{D^2\xi^i}{\partial v^2}.$$

Putting $\eta^i = \xi^i$ at P, so that $\phi = 0$, we have

$$\left(\frac{\partial\phi}{\partial v}\right)^2 = -\xi_i\frac{D^2\xi^i}{\partial v^2}. \tag{3.1}$$

Now, since $\xi_i\xi^i = 1$,

$$\xi_i\frac{D\xi^i}{\partial v} = 0 \quad \text{and} \quad \frac{D\xi_i}{\partial v}\frac{D\xi^i}{\partial v} + \xi_i\frac{D^2\xi^i}{\partial v^2} = 0,$$

and (3.1) may be written

$$\left(\frac{\partial\phi}{\partial v}\right)^2 = (\xi_v)^2. \tag{3.2}$$

Hence, from (2.9) and (3.2),

$$\begin{aligned}\left(\frac{\partial\phi}{\partial v}\right)^2 &= \Omega_{ij}\lambda^i\mu^j/g - \Omega_{12}/g, && \textit{in generalized } S_n,\\ &= R_{ijhk}\xi^i\mu^j\xi^h\mu^k - K, && \textit{in Riemannian } S_n,\\ &= -K, && \textit{in Euclidean } S_n.\end{aligned} \tag{3.3}$$

This gives the well-known result* that, for a Euclidean S_n, if S_2 is a developable and so $K = 0$, then $\partial\phi/\partial v = 0$, and a single plane is tangential to S_2 all along a generator.

Again points for which g vanishes must be excluded, for at such points the vector ξ^i is not defined.

* See (**2**), § 25.

4. Line of striction or extension

Let $t = t_0$ and $t = t_0 + \delta t$ be neighbouring generators intersected in points P and Q respectively by an orthogonal trajectory $v =$ constant, which is such that the arc-length PQ measured along the orthogonal trajectory is stationary for variation of v. If, as $\delta t \to 0$, the point $P \to (P)$, (P) is said to be a *central point* on the generator $t = t_0$. Further, if there exists a positive number η such that, provided $|\delta t| < \eta$, arc PQ is minimum (maximum) for variation of v, the central point (P) is said to be a *point of striction* (*extension*).

For variation of t_0 the locus of a central point will be called a *central line*, while the locus of a point of striction (extension) will be called a *line of striction* (*extension*).

If s denotes the arc-length measured along the orthogonal trajectory PQ, we have for ϵ, the modulus of the arc-length PQ, $\epsilon = |\delta t(\partial s/\partial t)_P| + O(\delta t^2)$. But, along $v =$ constant, $(\partial s/\partial t)^2 = g_{11} = g$. Hence

$$\left.\begin{aligned} \epsilon &= |\delta t|(\sqrt{g})_P + O(\delta t^2) \\ \frac{\partial \epsilon}{\partial v} &= |\delta t|\left(\frac{\partial \sqrt{g}}{\partial v}\right)_P + O(\delta t^2) \\ \frac{\partial^2 \epsilon}{\partial v^2} &= |\delta t|\left(\frac{\partial^2 \sqrt{g}}{\partial v^2}\right)_P + O(\delta t^2). \end{aligned}\right\} \tag{4.1}$$

Hence

A necessary and sufficient condition that a point be a central point on a given generator is that $\partial \sqrt{g}/\partial v = 0$ *at the point.* (4.2)

Since the equation $\partial \sqrt{g}/\partial v = 0$ may have more than one root, there may be more than one central point on a generator. Further,

A sufficient condition that a central point be a point of striction (*extension*) *is that* $\partial^2 \sqrt{g}/\partial v^2$ *be positive* (*negative*) *at the point.* (4.3)

Now
$$g = g_{11} = g_{\alpha\beta}\lambda^\alpha \lambda^\beta = g_{ij}\lambda^i \lambda^j,$$

and
$$\frac{\partial g}{\partial v} = \frac{D}{\partial v}(g_{ij}\lambda^i \lambda^j) = 2\lambda_i \frac{D\lambda^i}{\partial v},$$

and so
$$\frac{\partial g}{\partial v} = 2\lambda_i \frac{D\mu^i}{\partial t} \tag{4.4}$$

from (1.13). Then

$$\frac{\partial^2 g}{\partial v^2} = 2\frac{D}{\partial v}\left(\lambda_i \frac{D\mu^i}{\partial t}\right) = 2\frac{D\lambda_i}{\partial v}\frac{D\mu^i}{\partial t} + 2\lambda_i \frac{D^2\mu^i}{\partial v \partial t}.$$

But $\lambda^i = \sqrt{g}\xi^i$, and so $D\lambda^i/\partial v = \sqrt{g}D\xi^i/\partial v$ at a central point since then $\partial\sqrt{g}/\partial v = 0$. Thus

$$2\frac{D\lambda_i}{\partial v}\frac{D\mu^i}{\partial t} = 2\frac{D\lambda_i}{\partial v}\frac{D\lambda^i}{\partial v} = 2g\frac{D\xi_i}{\partial v}\frac{D\xi^i}{\partial v} = 2g(\xi_v)^2.$$

Further,

$$\frac{D^2\mu^i}{\partial v\partial t} = \left[\frac{D^2}{\partial v\partial t}\right]\mu^i = \Omega_j{}^i\mu^j$$

since $D\mu^i/\partial v = 0$ from (ii), so that

$$\frac{\partial^2 g}{\partial v^2} = 2g(\xi_v)^2 - 2\Omega_{ij}\lambda^i\mu^j,$$

and, from (2.8),

$$\frac{\partial^2 g}{\partial v^2} = -2\Omega_{12} \quad \textit{if} \quad \frac{\partial\sqrt{g}}{\partial v} = 0. \tag{4.5}$$

Now
$$\frac{\partial\sqrt{g}}{\partial v} = \frac{1}{2\sqrt{g}}\frac{\partial g}{\partial v},$$

and so
$$\frac{\partial\sqrt{g}}{\partial v} = \xi_i\frac{D\mu^i}{\partial t} \tag{4.6}$$

from (4.4), provided that $g \neq 0$. We shall henceforth consider only points for which $g \neq 0$, for, if $g = 0$, the vector ξ^i is not defined. Also

$$\frac{\partial^2\sqrt{g}}{\partial v^2} = \frac{1}{2\sqrt{g}}\frac{\partial^2 g}{\partial v^2} \quad \text{if} \quad \frac{\partial\sqrt{g}}{\partial v} = 0$$

and so
$$\frac{\partial^2\sqrt{g}}{\partial v^2} = -\frac{\Omega_{12}}{\sqrt{g}} \quad \text{if} \quad \frac{\partial\sqrt{g}}{\partial v} = 0 \tag{4.7}$$

from (4.5). We may now rewrite (4.2) and (4.3), using (4.6) and (4.7), as

A necessary and sufficient condition that a point be a central point on a given generator is that

$$\xi_i D\mu^i/\partial t = 0 \textit{ at the point.}$$

A sufficient condition that a central point be a point of striction (extension) is that (4.8)

in generalized S_n, Ω_{12} be $-(+)$ at the point;

in Riemannian or Euclidean S_n, K be $-(+)$ at the point.

If ν^i is the unit vector tangential to any curve C on S_2 intersecting the generators at an angle θ, which we shall suppose does not vanish at any point of C, then $\cos\theta = \nu^i\mu_i$. For variation of t and v,

$$-\sin\theta\, d\theta = D\nu^i\mu_i + \nu_i\left(\frac{D\mu^i}{\partial v}dv + \frac{D\mu^i}{\partial t}dt\right)$$

$$= D\nu^i\mu_i + \nu_i\frac{D\mu^i}{\partial t}dt,$$

whilst $\nu^i = \mu^i\cos\theta + \xi^i\sin\theta$. Hence

$$-\sin\theta\, d\theta = \mu_i D\nu^i + dt\sin\theta\,\xi_i\frac{D\mu^i}{\partial t}.$$

Thus, if two of the expressions $d\theta$, $\mu_i D\nu^i$, $\xi_i(D\mu^i/\partial t)$ vanish, so does the third. Hence

If a curve on S_2 possesses two of the following properties, it also possesses the third:

(1) *it cuts the generators at a constant angle;* (4.9)

(2) *it is autoparallel in S_n, or its principal normals are orthogonal to the generators;*

(3) *it is a central line.*

This is the generalization of a theorem of Bonnet's.*

REFERENCES

1. E. T. Davies, 'Metric spaces based on a vector density': *Proceedings of the London Math. Soc.* (in course of publication).
2. L. P. Eisenhart, *A Treatise on the Differential Geometry of Curves and Surfaces* (1909).
3. J. G. Freeman, 'Generalized metric spaces': London Univ. Thesis (1943).
4. —— 'First and second variations of the length integral in a generalized metric space': *Quart. J. of Math.* (Oxford), 15 (1944), 70–83.
5. J. A. Schouten and J. Haantjes, 'Über die Festlegung von allgemeinen Massbestimmungen und Übertragungen in Bezug auf ko- und kontravariante Vektordichten': *Monats. für Math. und Phys.* 43 (1936), 161–76.

* See (2), § 106.

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